UNIVERSIDADE FEDERAL DO CEARÁ
DEPARTAMENTO DE COMPUTAÇÃO
CURSO DE DOUTORADO EM COMPUTAÇÃO

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PARÂMETROS DE COLORAÇÃO DE VÉRTICES BASEADOS EM ALGORITMOS DE COLORAÇÃO

FORTALEZA, CEARÁ

2011

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Tese submetida à Coordenação do Curso de Doutorado em Ciência da Computação da Universidade Federal do Ceará, como requisito parcial para a obtenção do grau de Doutor em Ciência da Computação.

Área de concentração: Teoria dos Grafos

Orientador: Cláudia Linhares Sales

FORTALEZA, CEARÁ

2011

CAMPOS, V. A..

Parâmetros de coloração de vértices baseados em algoritmos de coloração / Victor Almeida Campos. 2011.

131p.;il. color. enc.

Orientador: Cláudia Linhares Sales

Tese(Doutorado em Computação) - Universidade Federal do Ceará, Departamento de Computação, Fortaleza, 2011.

1. Coloração de vértices 2. Número de Grundy 3. Número b-cromático I. Cláudia Linhares Sales(Orient.) II. Universidade Federal do Ceará– Doutorado em Computação(Doutorado) III. Doutor

AGRADECIMENTOS

Primeiro de tudo, gostaria de agradecer à Cláudia Linhares Sales, minha orientadora. Sua ajuda foi crucial para transformar a minha escrita tosca em um texto que estou com orgulho de submeter. Também agradeço por todo o apoio me levando para congressos, workshops e me apresentando colaboradores para conseguir novos problemas de pesquisa. Como se isto não fosse o suficiente, ainda me passa dicas de vez em quando de ética entre pesquisadores para eu deixar de fazer besteira por aí.

Um agradecimento especial à banca que aceitou ler minha tese com menos de um mês de aviso. Muito obrigado a Frédéric Havet, Jayme Szwarcfiter, Cláudio Lucchesi e Rudini Sampaio.

Agradeço a todos os meus coautores. Trabalhar com cada um de vocês foi uma oportunidade única de aprendizado. Especialmente com os ótimos coautores que tive até hoje.

Esta tese é fruto tanto de muito trabalho duro quanto de bons professores e bons colegas que conseguiram transmitir seus conhecimentos com grande clareza. Devo isto a muitos, mas mando um agradecimento especial a todos os meus professores em Montreal. Agradeço ao Luc Devroye que me recebeu tão bem e me serve como referencial de como apresentar uma aula de matemática. A clareza do seu conhecimento e das suas explicações me ajudaram e me ajudam até hoje. Também agradeço ao Luc por tudo que fez por mim em Montreal e especialmente à sua amizade. Agradeço ao Vašek Chvátal pelas suas maravilhosas histórias e aulas sobre Paul Erdős e por ter me convidado para o seu grupo especial de pesquisa ConCoCO. Agradeço ao Bruce Reed por ter me aceitado como aluno e ter me levado aos seus workshops em Barbados e Redonda para trabalhar com pesquisadores de tão alto nível. Você foi um grade amigo até mesmo nas horas mais difíceis antes de voltar a Fortaleza. Muito obrigado. Agradeço também ao Bruce Shepherd por acreditar em mim quando mal nos conhecíamos e me convidar para o seu workshop em Barbados.

A McGill é um excelente ambiente para o crescimento de um pesquisador não apenas pelos ótimos professores mas também pelos ótimos colegas e amigos. Agradeço especialmente a companhia da minha colega de sala e grande amiga Perouz Taslakian. Agradeço também ao grupo de almoço. “Lunch!!!” Um abraço especial também para todos os colegas que conheci por lá, especialmente ao Imad, Jon, Patrick, Joe e Bob que sempre estavam disponíveis para beber depois de semanas tão cansativas. Agradeço também ao Benelux por nos forcener o ambiente para isto.

Agradeço a todos do grupo ParGO que acreditaram que eu conseguiria algum dia e me ajudam até hoje. Um agradecimento especial aos jurássicos Manoel, Ricardo e Cláudia que começaram esta bola de neve. Agradeço também pela casa espaçosa do Manoel. Mesmo que nunca tenha visitado, sei que algum dia serei convidado para participar das partidas de futebol no campo do tamanho oficial da FIFA e depois relaxar em uma das suas três saunas aromatizadas.

Agradeço também à minha família. Devo tudo a eles pelo apoio por tantos anos. Em especial a minha mãe, à Shibinha, à Michelle e ao Davi. Também agradeço à Lola pelo toque de marrom na minha vida. Agradeço a todos os buttless pela sua amizade, jogatina e bebedeira. Agradeço ao Marcelo Lobisomem pela sua amizade e pela sua enorme força capaz de parar até mesmo o movimento de rotação da terra, se assim ele desejar. Também agradeço por ele não ter feito isso ainda.

Utilizando este espaço com liberdade para dizer o que sinto, aproveito para não agradecer aos novos salários dos políticos no Brasil. “Isto é uma vergonha!”

"Se queres conhecer um homem, não escute suas palavras, mas veja suas ações."

(Rômulo Luz)

RESUMO

Uma coloração de vértices de um grafo G é uma atribuição de cores aos vértices de G de forma que vértices adjacentes recebam cores distintas. O problema de coloração de vértices consiste em achar o menor número de cores em uma coloração de vértices de G. Tal número é chamado de número cromático de G e é representado por $\chi(G)$. Devido à dificuldade de achar $\chi(G)$ para grafos em geral, estudamos dois algoritmos polinomiais que geram colorações que não têm necessariamente o menor número de cores. Em se tratando de heurísticas, estudamos o pior desempenho desses algoritmos.

O primeiro algoritmo, o algoritmo guloso, recebe os vértices do grafo em uma ordem qualquer e atribui a cada vértice a menor cor ainda não utilizada em seus vizinhos. O segundo algoritmo, o algoritmo de b-coloração, recebe uma coloração qualquer do grafo e, uma a uma, sempre que possível elimina uma cor utilizada, recolorindo cada vértice daquela cor com uma cor não utilizada em qualquer um dos seus vizinhos. As colorações produzidas pelo primeiro e segundo algoritmo chamam-se coloração gulosa e b-coloração, respectivamente. Para um grafo G, o maior número de cores utilizadas em uma coloração gulosa e b-coloração é denotado por $\Gamma(G)$ e $\chi_b(G)$, respectivamente.

Nesta tese, apresentamos novos limites para $\Gamma(G[H])$, $\Gamma(G\square H)$, $\Gamma(G\times H)$, $\Gamma(G\boxtimes H)$ em função de $\Gamma(G)$, $\Gamma(H)$, $\Delta(G)$ e $\Delta(H)$, onde $G[H]$, $G\square H$, $G\times H$ e $G\boxtimes H$ são os grafos obtidos pelo produto lexicográfico, cartesiano, direto e forte dos grafos G e H, respectivamente. Também apresentamos algoritmos polinomiais para encontrar o número b-cromático de grafos $(q,q-4)$ e cactos com $m(G)\geq 7$, onde $m(G)$ é um limite superior para $\chi_b(G)$ igual ao maior inteiro k tal que G possui k vértices, cada um com grau pelo menos $k-1$. No caso em que G é um cacto, provamos também que $\chi_b(G)$ é igual a $m(G)$ ou $m(G)-1$.

Palavras-chave: Coloração de vértices. Número de Grundy. Número b-cromático.

ABSTRACT

A vertex colouring in a graph G is a mapping from the set of vertices of G to a set of colours such that no two adjacent vertices get the same colour. The vertex colouring problem consists in finding the least number of colours in a vertex colouring of G. This number is called the chromatic number of G and is denoted by $\chi(G)$. Given the difficulty in finding $\chi(G)$ for general graphs, we study two polinomial time algorithms that obtain colourings of G that are not guaranteed to be optimal. For an heuristic analysis, we consider the worst colouring obtained by these algorithms.

The first algorithm, the greedy algorithm, considers an order of the vertices mapping each vertex iteratively to the smallest colour not used in one of its neighbours. The second algorithm, the b-colouring algorithm, considers a colouring of the vertices and, one by one, as long as it is still possible reduces a colour class by recolouring each vertex with that colour to a colour not used in one of it's neighbours. The colourings produced by the first and second algorithm are called greedy colouring and b-colouring, respectively. Given a graph G, the maximum number of colours used in a greedy colouring and a b-colouring is denoted by $\Gamma(G)$ and $\chi_b(G)$, respectively.

In this thesis, we present new bounds for $\Gamma(G[H])$, $\Gamma(G\square H)$, $\Gamma(G\times H)$, $\Gamma(G\boxtimes H)$ as a function of $\Gamma(G)$, $\Gamma(H)$, $\Delta(G)$ and $\Delta(H)$, where $G[H]$, $G\square H$, $G\times H$ and $G\boxtimes H$ are the graphs obtained through the lexicographic, cartesian, direct and strong products of G and H, respectively. We also present polinomial time algorithms to find the b-chromatic number of $(q,q-4)$ and cacti graphs with $m(G)\geq 7$, where $m(G)$ is an upper bound to $\chi_b(G)$ equal to the largest k for which G has k vertices with degree at least $k-1$ each. When G is a cactus, we also prove that $\chi_b(G)$ is either $m(G)$ or $m(G)-1$.

Keywords: Vertex colouring. Grundy number. b-chromatic number.

SUMÁRIO

1 INTRODUÇÃO

Dado um inteiro positivo k, uma *coloração* ou k-*coloração* de um grafo $G = (V,E)$ é uma atribuição de cores do conjunto $\{1,\ldots,k\}$ aos vértices de G tal que cada vértice recebe uma cor e vértices adjacentes recebem cores diferentes. O problema de coloração de vértices é definido como o problema de encontrar o menor valor $\chi(G)$, conhecido como número cromático de G, tal que G admite uma $\chi(G)$-coloração. Um *conjunto independente* de G é um subconjunto de seus vértices não adjacentes dois a dois. Chamamos o conjunto de vértices que recebem uma mesma cor de *classe de cor*. Como cada classe de cor é um conjunto independente, uma k-coloração também pode ser vista como uma partição de V em k conjuntos independentes. Neste caso, o número cromático de G é o menor número de conjuntos independentes necessários para particionar V. Um grafo G é dito k-*colorível* se G admite uma k-coloração.

O problema de coloração de vértices é um dos modelos mais estudados em Teoria dos Grafos pela sua relevância, em campos práticos e teóricos. Do ponto de vista teórico, o problema de coloração de vértices é de fundamental importância para a matemática discreta, pois costuma aparecer regularmente em problemas sem conexão aparente com coloração. Um bom exemplo ocorre no Teorema de Erdős-Stone-Simonovits [1] que mostra que, para um grafo fixo G, o número máximo de arestas $f(n,G)$ em um grafo com n vértices que não contém G depende do número cromático de G:

$$\lim_{n\to\infty}\frac{f(n,G)}{n^2} = \frac{\chi(G)-2}{2\chi(G)-2}.$$

Um resultado clássico [2, 3], e de fato um dos primeiros, de Teoria da Complexidade diz que determinar se um grafo é k-colorível é um problema NP-Completo, para qualquer inteiro $k \geq 3$. Para $k = 3$, o problema permanece NP-Completo mesmo se considerarmos grafos sem triângulos e grau máximo quatro [4]. Algoritmos aproximativos também não são promissores pois, a menos que $P = NP$, não existe algoritmo polinomial que aproxime o número cromático por um fator constante [5].

Coloração de grafos também é reconhecida pelas suas aplicações, uma vez que corresponde ao problema fundamental de particionar um conjunto de objetos em classes, de acordo com regras pré-estabelecidas. Escalonamento de tarefas [6], alocação de frequências [7], alocação de registradores [8, 9] e métodos de elementos finitos [10] em suas diversas formas têm problemas de coloração em sua natureza.

A teoria de coloração de grafos tem despertado a atenção de pesquisadores há muitos anos. Para mostrar este interesse, basta observar uma sequência de conferências internacionais com o único objetivo de apresentar os avanços recentes na área [11], além da publicação de um livro apresentando vários problemas de coloração de grafos, resultados e as relações entre eles [12].

Muitos limitantes superiores para o número cromático surgem de algoritmos que produzem colorações. Um dos mais conhecidos é o algoritmo guloso. Uma coloração gulosa relativa à ordem $v_1 < v_2 < \cdots < v_n$ de V é obtida ao colorir os vértices em ordem atribuindo a v_i o menor inteiro positivo não utilizado nos seus vizinhos em $\{v_1, \ldots, v_{i-1}\}$. Christen e Selkow [13] definiram o *número de Grundy* $\Gamma(G)$ como o maior número de cores de uma coloração gulosa de G. Pela forma em que os vértices são coloridos podemos concluir que $\Gamma(G) \leq \Delta(G) + 1$, onde $\Delta(G)$ denota o grau máximo de um vértice de G.

Outro algoritmo estudado nesta tese tem como saída uma b-coloração. Este algoritmo recebe como entrada uma coloração qualquer do grafo. Em seguida, enquanto for possível, o algoritmo tenta reduzir a quantidade de cores utilizadas da seguinte maneira: se existe uma classe de cor em que cada vértice não possui vizinhos em alguma outra classe de cor, então podemos recolorir os vértices desta classe para obter uma coloração que usa menos cores. Irvin e Manlove [14] definiram o número b-cromático $\chi_b(G)$ como o maior número de cores de uma b-coloração de G. Em cada classe de cor de uma b-coloração com k cores, existe pelo menos um vértice adjacente às $k-1$ outras cores. Se $m(G)$ é o maior inteiro tal que G tem pelo menos $m(G)$ vértices com grau $m(G)-1$ cada, então $\chi_b(G) \leq m(G)$.

Devido à dificuldade do problema de coloração de vértices para um grafo qualquer, estudos costumam analisar o problema reduzido a algumas classes de grafos. De fato, o problema se torna fácil se nos restringirmos a algumas classes específicas de grafos. Uma classe de grafos bastante estudada sob o aspecto de coloração foi a classe de grafos perfeitos. Um grafo é perfeito se todos os seus subgrafos induzidos tem número cromático igual ao tamanho do seu maior subgrafo completo. O estudo desta classe de grafos gerou um livro com resultados relacionados [15] e exemplifica o interesse em estudar o problema de coloração para classes restritas de grafos.

Mesmo a coloração de vértices sendo polinomial para grafos perfeitos [16], isto não se mantém para as variações de coloração estudadas nesta tese. Por exemplo, é NP-completo verificar se $\Gamma(G) = \Delta(G) + 1$ [17] ou se $\chi_b(G) = m(G)$ [18] em grafos bipartidos. Obviamente, não podemos esperar que os algoritmos guloso e de b-coloração obtenham uma coloração com o número mínimo de cores, dado que é NP-Difícil achar o número cromático de G. O objetivo desta tese é fazer uma análise do pior caso destes algoritmos ao analisar os parâmetros $\Gamma(G)$ e $\chi_b(G)$ para algumas classes de grafos.

No Capítulo 2, apresentamos as definições e resultados gerais que serão úteis nos capítulos subsequentes. No Capítulo 3, estudamos o número de Grundy de produtos de grafos. No Capítulo 4, mostramos como calcular o número b-cromático de grafos $(q, q-4)$. No Capítulo 5, mostramos como calcular o número b-cromático de cactos. Cada capítulo apresenta um resumo estendido e discussões dos resultados obtidos. As demonstrações completas dos resultados contidos nesta tese se encontram nos anexos. As demonstrações relativas ao Capítulo 3 e ao Capítulo 4 se encontram no Anexo A e B, respectivamente. As demonstrações relativas ao Capítulo 5 se encontram no Anexo C e D. No Anexo E, colocamos o resumo de outros resultados obtidos durante o doutorado que não foram incluídos nesta tese pela baixa aderência ao foco principal.

2 NOTAÇÕES E TERMINOLOGIA

Um *grafo* é um par ordenado (V,E) onde V é um conjunto de *vértices* e E é um conjunto de *arestas*. Os conjuntos V e E são disjuntos e cada aresta corresponde a um par não ordenado de vértices. Se existe uma aresta e entre vértices u e v, representamos esta aresta por uv ou vu, dizemos que e *incide* em u e v e que u e v são *vizinhos*. Dizemos que u e v são as *extremidades* de uv e que u e v são *adjacentes*. Dizemos que duas arestas são *adjacentes* se elas compartilham pelo menos uma extremidade.

Seja G um grafo. Utilizamos $V(G)$ e $E(G)$ para representar os conjuntos de vértices e arestas de G, respectivamente. Dizemos que G é *finito* se $V(G)$ e $E(G)$ são finitos. Um *laço* em G é uma aresta cujas extremidades são iguais e dizemos que G tem *arestas múltiplas* se G tem duas arestas distintas com as mesmas extremidades. Os grafos considerados nesta tese são finitos e não contém laços ou arestas múltiplas.

Seja $v \in V(G)$. A *vizinhança* de v em G é o conjunto $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. A *vizinhança fechada* de v é o conjunto $N_G[v] = N_G(v) \cup \{v\}$. O *grau* de v é o número de arestas incidentes em v e utilizamos $d_G(v)$ para denotar o grau de v. Usamos $\Delta(G)$ para o grau máximo de um vértice de G. Quando não houver ambiguidade, omitimos G destes parâmetros e utilizamos V, E, $N(v)$, $N[v]$, $d(v)$ e Δ. Se $X \subseteq V(G)$, então $N^X(v)$ representa o conjunto $N(v) \cap X$.

Um *subgrafo* de G é um grafo H tal que $V(H) \subseteq V(G)$ e $E(H) \subseteq E(G)$. Dizemos que H é *subgrafo induzido* de G se, para todo $u,v \in V(H)$, se $uv \in E(G)$ então $uv \in E(H)$. Para $X \subseteq V(G)$, o *subgrafo de G induzido por X* é o subgrafo induzido de G com conjunto de vértices X.

Um *caminho* em G é uma sequência de vértices $P = v_1v_2\cdots v_q$ onde $v_iv_{i+1} \in E(G)$ para $i = 1,\ldots,q-1$ e $v_i \neq v_j$ para todo $1 \leq i \neq j \leq q$. Dizemos que v_1 e v_q são as *extremidades* de P e, para $i \in \{2,\ldots,q-1\}$, dizemos que v_i é um vértice interno de P. Este caminho é um *ciclo* se $v_1v_q \in E(G)$. O *tamanho* de um caminho ou ciclo é o número de arestas no caminho ou ciclo. Uma *corda* em um caminho ou ciclo é qualquer aresta entre seus vértices não utilizada para definir este caminho ou ciclo. Um *caminho induzido* é um caminho sem cordas e um *ciclo induzido* é um ciclo sem cordas.

A *distância* entre dois vértices u e v é o tamanho do menor caminho contendo u e v como extremidades. O *raio* de G é o menor inteiro r de forma que existe um vértice c de G tal que a distância de qualquer outro vértice para c é no máximo r.

Dizemos que G é *conexo* se existe um caminho entre qualquer par de vértices de G. Se G não é conexo, dizemos que ele é *desconexo*. Uma *componente* de G é um subgrafo conexo maximal de G. O *complemento* de G é o grafo $\overline{G}$ com $V(\overline{G}) = V(G)$ e $uv \in E(\overline{G})$ se, e somente se, $uv \notin E(G)$.

Um grafo é uma *árvore* se ele é conexo e não possui ciclos. Um grafo é *completo* se os seus vértices são dois a dois adjacentes e *vazio* se os seus vértices são dois a dois não adjacentes. Um grafo é *bipartido* se o conjunto de vértices pode ser particionado em dois subconjuntos X e Y tal que toda aresta tem uma extremidade em X e a outra em Y. Tal partição (X,Y) é chamada a *bipartição* do grafo e X e Y as suas *partes*. Um grafo é *bipartido completo* se todo vértice em X é adjacente a todo vértice em Y da sua bipartição (X,Y).

Um *conjunto independente* é um conjunto de vértices não adjacentes dois a dois e uma *clique* é um conjunto de vértices adjacentes dois a dois. Um *emparelhamento* é um conjunto de arestas duas a duas não adjacentes.

Denote por K_n o grafo completo com n vértices e por $K_{p,q}$ o grafo bipartido completo com partes de tamanho p e q. Denote por S_n o grafo sem arestas com n vértices e por P_k o grafo que consiste em um caminho com k vértices.

2.1 Produto de grafos

Dados dois grafos G e H, o *produto direto* $G \times H$, o *produto lexicográfico* $G[H]$, o *produto cartesiano* $G \square H$ e o *produto forte* $G \boxtimes H$ são os grafos com conjunto de vértices $V(G) \times V(H)$ e com os seguintes conjuntos de arestas:

$$
\begin{aligned}
E(G \times H) &= \{(a,x)(b,y) \mid ab \in E(G) \text{ e } xy \in E(H)\}; \\
E(G[H]) &= \{(a,x)(b,y) \mid ab \in E(G) \text{ ou } (a = b \text{ e } xy \in E(H))\}; \\
E(G \square H) &= \{(a,x)(b,y) \mid (a = b \text{ e } xy \in E(H)) \text{ ou } (ab \in E(G) \text{ e } x = y)\}; \\
E(G \boxtimes H) &= E(G \times H) \cup E(G \square H).
\end{aligned}
$$

Podemos visualizar estes conjuntos de arestas de uma forma mais intuitiva como descrevemos a seguir. No produto lexicográfico, criamos uma cópia de H para cada vértice de G. Em seguida, colocamos todas as arestas entre duas cópias de H sempre que os vértices correspondentes em G são adjacentes. Já para construir os produtos cartesiano, direto e forte utilizamos uma regra em comum. Considere os vértices em $V(G) \times V(H)$ dispostos em uma matriz indexada nas linhas por vértices de H e nas colunas por vértices de G. Para cada par de arestas $ab \in E(G)$ e $xy \in E(H)$, colocamos arestas entre os vértices relacionados no produto para representar um $\square$ no produto cartesiano, um $\times$ no produto direto e um $\boxtimes$ no produto forte. Na Figura 2.1, exemplificamos os produtos $G[H]$, $G \square H$, $G \times H$ e $G \boxtimes H$ no caso em que G e H são isomorfos a um P_3.

2.2 Decomposição primeval

Dizemos que G é um *cografo* se G não possui nenhum P_4 induzido [19]. Dizemos que G é um *grafo P_4-esparso* se qualquer conjunto de cinco vértices induz no máximo um P_4. Grafos P_4-esparsos foram introduzidos em [20], abrangem os cografos e podem ser reconhecidos em tempo linear [21].

A *união* de dois grafos G_1 e G_2 é o grafo $G_1 \cup G_2$, onde $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ e $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A *junção* de dois grafos G_1 e G_2 é o grafo $G_1 \vee G_2$,

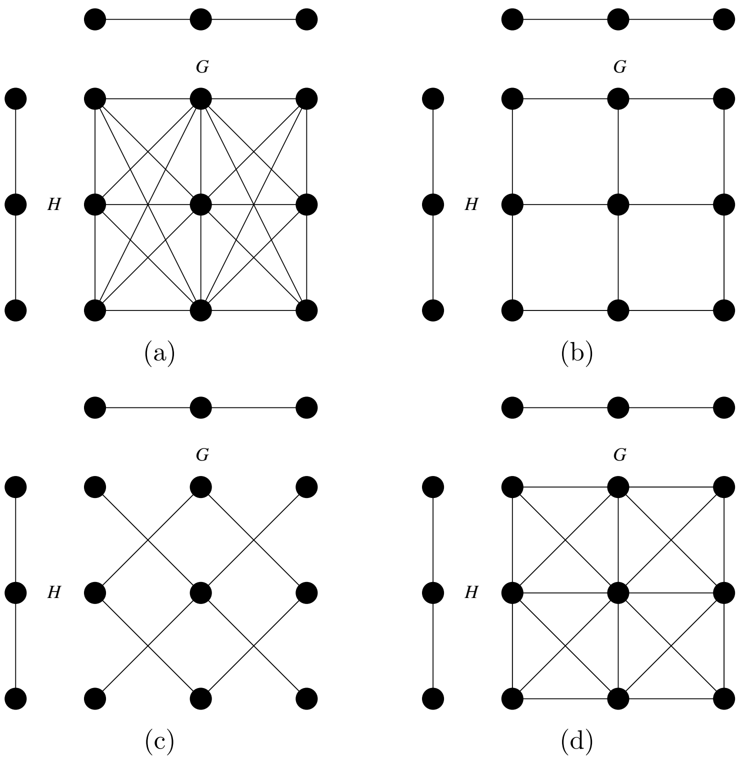


Figura 2.1: Produtos entre dois grafos isomorfos ao P_3.

onde $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ e

$$E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup V(G_1) \times V(G_2).$$

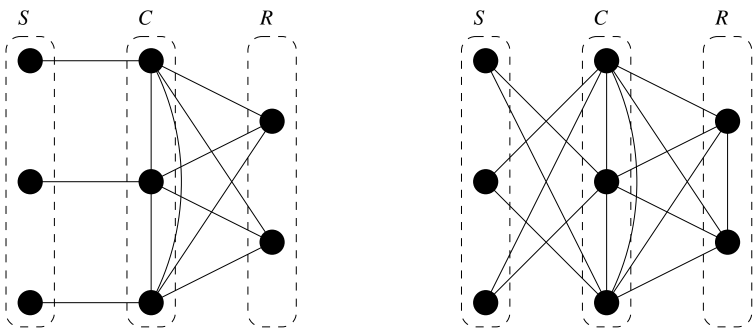
A operação de junção consiste na operação de união com a inclusão de todas as arestas entre os vértices de G_1 e G_2.

Teorema 2.1 ([19]). *Se G é um cografo com pelo menos dois vértices, então existem cografos G_1 e G_2, subgrafos de G, tal que G é igual à união ou à junção de G_1 e G_2.*

Dizemos que um grafo é uma *aranha* se o seu conjunto de vértices pode ser particionado em (S,C,R) onde $S = \{s_1,\ldots,s_k\}$ e $C = \{c_1,\ldots,c_k\}$ para algum $k \geq 2$ formam, respectivamente, um conjunto independente e uma clique; s_i é adjacente a c_j se e só se $i = j$ (*aranha magra*), ou s_i é adjacente a c_j se e só se $i \neq j$ (*aranha gorda*); e todo vértice de R é adjacente a todo vértice de C e não-adjacente a nenhum vértice de S. Chamamos os conjuntos S, C e R de *"pernas"*, *"corpo"* e *"cabeça"* da aranha, respectivamente. Dizemos que a aranha é sem cabeça se $R = \emptyset$. Na Figura 2.2, mostramos exemplos de uma aranha magra e de uma aranha gorda indicando os conjuntos S, C e R. Observe que um grafo é o complemento do outro e que o complemento de uma aranha magra é uma aranha gorda.

Teorema 2.2 ([20, 22]). *Se G é um grafo P_4-esparso com pelo menos dois vértices, então um dos seguintes é válido para G:*

1. *G é a união de dois grafos P_4-esparsos;*



(a) magra (b) gorda

Figura 2.2: Exemplo de grafos aranha.

2. $\overline{G}$ é a junção de dois grafos P_4-esparsos; ou

3. G é uma aranha e a sua cabeça é um grafo P_4-esparso.

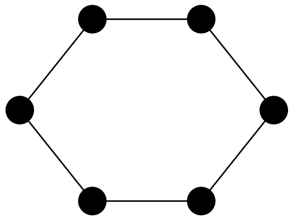
Os Teoremas 2.1 e 2.2 nos fornecem decomposições conhecidas para cografos e grafos P_4-esparsos. Uma *árvore de decomposição* de um grafo G é uma árvore T_G onde cada nó interno é rotulado com uma operação sobre grafos e representa um subgrafo de G obtido através da aplicação dessa operação nos seus filhos. A raiz de T_G representa o grafo original G e suas folhas representam grafos básicos.

O Teorema 2.1 implica que qualquer cografo G possui uma árvore de decomposição T_G com apenas duas operações (união e junção) e cujas folhas representam os vértices de G. Essa árvore pode ser calculada em tempo polinomial [23].

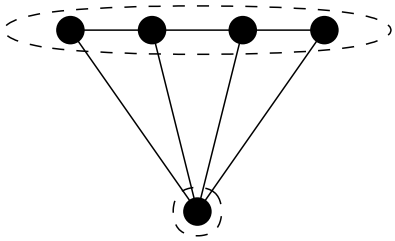
O Teorema 2.2 implica que qualquer grafo P_4-esparso G possui uma árvore de decomposição T_G com três operações (união, junção e operação aranha) cujas folhas representam os vértices de G [22]. Essa árvore também pode ser calculada em tempo polinomial [21].

Um grafo G é *p-conexo* se, para toda partição de $V(G)$ em dois conjuntos não vazios A e B, existe um P_4 induzido com vértices em A e B. Um grafo p-conexo G é *separável* se existe uma partição de $V(G)$ em subconjuntos A e B não-vazios tal que cada P_4 induzido que possui vértices de A e B tem seus pontos internos em A e suas extremidades em B. Um subgrafo induzido que é p-conexo e, além disso, é maximal com relação a essa propriedade, é chamado de *p-componente*. Observe que se G tem apenas um vértice, então G é p-conexo por vacuidade. Assim, as p-componentes de G tem um ou pelo menos quatro vértices. Na Figura 2.3, mostramos o exemplo de um grafo p-conexo e de um grafo não p-conexo. No caso do grafo que não é p-conexo, circulamos as suas p-componentes.

Para obter as p-componentes de G, considere o grafo G_p construído da seguinte forma. O conjunto de vértices de G_p é o mesmo conjunto de vértices de G. Se $u, v \in V(G)$ e existe um P_4 induzido em G que contém ambos u e v, adicionamos a aresta uv em G_p. As p-componentes de G correspondem às componentes conexas de G_p. Observe que o complemento de um P_4 também é um P_4. Assim, as componentes de G e de $\overline{G}$ são as



(a) p-conexo



(b) p-componentes circuladas

Figura 2.3: Exemplo de grafos aranha.

mesmas.

O teorema abaixo de [24] é um importante teorema estrutural para grafos gerais usando grafos p-conexos.

Teorema 2.3 ([24])**.** *Dado um grafo G, exatamente uma das seguintes afirmações é válida:*

- *G é a união de dois subgrafos induzidos disjuntos;*
- *$\overline{G}$ é a junção de dois subgrafos induzidos disjuntos;*

(iii) G contém uma única p-componente separável própria H com partição (H_1, H_2) tal que todo vértice que está fora de H é adjacente a todo vértice de H_1 e a nenhum vértice de H_2;

(iv) G é p-conexo.

O Teorema 2.3 sugere uma decomposição para grafos em geral, chamada *decomposição primeval*, que pode ser calculada em tempo polinomial [24]. As folhas da árvore da decomposição primeval de um grafo são as suas componentes p-conexas (ver (iv)). Os nós internos são rotulados com as operações *união* (ver (i)), *junção* (ver (ii)) ou a operação definida por (iii), que chamaremos *operação 3*. Se H é p-conexo separável com partição (H_1, H_2) e G' é um grafo disjunto em vértices de H, a operação 3 adiciona arestas entre estes grafos tornando todo vértice de G' adjacente a todo vértice de H_1 (partição que possui os vértices internos dos P_4's que o cruzam) e a nenhum vértice de H_2.

2.3 Coloração de grafos

Uma *k-coloração* (*própria*) de um grafo G é uma função $\psi : V(G) \to \{1, \dots, k\}$ tal que, para cada aresta $uv \in E(G)$, $\psi(u) \neq \psi(v)$. Uma k-coloração também pode ser vista como uma partição do conjunto de vértices de G em k conjuntos independentes não vazios $C_i = \{v \mid \psi(v) = i\}$ para $1 \leq i \leq k$. Por conveniência, com um certo abuso de terminologia, por uma k-coloração, referimos tanto à função ψ como à partição $(C_1, \dots, C_k)$. Os elementos em $\{1, \dots, k\}$ são chamados de *cores* e os elementos em $\{C_1, \dots, C_k\}$ são chamados de *classes de cor*. Um grafo é k-*colorível* se ele admite uma k-coloração. O *número cromático* $\chi(G)$ é o menor inteiro k tal que G é k-colorível.

Uma *coloração gulosa* relativa a uma ordem $v_1 < v_2 < \cdots < v_n$ dos vértices de G é obtida ao colorir os vértices em ordem, atribuindo a v_i o menor inteiro positivo não utilizado nos seus vizinhos com índice menor. Em uma coloração gulosa, para todo $i < j$, todo vértice em C_j tem um vizinho em C_i pois, caso contrário, tal vértice em C_j receberia uma cor menor do que ou igual a i. Em contrapartida, qualquer coloração que satisfaz esta propriedade é uma coloração gulosa para qualquer ordem em que os vértices de C_i precedem os vértices de C_j sempre que $i < j$. O *número de Grundy* $\Gamma(G)$ é o maior inteiro k tal que G tem uma k-coloração gulosa.

Uma *árvore binomial* é uma árvore T_k definida recursivamente da seguinte forma. Para $k = 1$, T_1 é a árvore enraizada contendo apenas um vértice. Para $k \geq 2$, T_k é obtida a partir de T_{k-1} ao adicionar, para cada vértice v de T_{k-1}, um vértice v' e uma aresta vv'. Na árvore T_k, mantemos a mesma raiz da árvore T_{k-1}. Esta árvore é importante para estudar o número de Grundy pois T_k é a menor árvore com número de Grundy k [25, 26]. Na Figura 2.4, exibimos a árvore binomial T_5 e uma coloração gulosa de seus vértices com 5 cores.

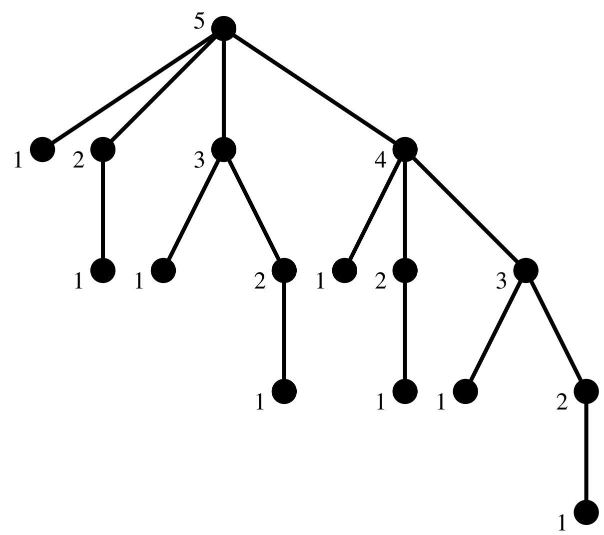


Figura 2.4: A árvore binomial T_5.

Seja ψ uma k-coloração de G. Dizemos que $v \in V(G)$ é um *b-vértice* em ψ se v é adjacente a pelo menos um vértice colorido com cada cor diferente de $\psi(v)$. Dizemos que ψ é uma *b-coloração* se cada classe de cor de ψ possui pelo menos um b-vértice. O *número b-cromático* $\chi_b(G)$ é o maior inteiro k tal que G possui uma b-coloração com k cores [14]. Naturalmente, uma coloração com $\chi(G)$ cores é uma b-coloração, dado que esta não pode ser melhorada. Assim, $\chi_b(G)$ é bem definido.

Seja G um grafo e ψ uma b-coloração de G com k cores. Para $1 \leq i \leq k$, seja v_i um b-vértice que recebe a cor i. Dizemos que $\{v_1, \ldots, v_k\}$ é uma *base* da b-coloração ψ. Note que uma b-coloração pode ter várias bases.

Como qualquer b-coloração de G é uma coloração própria, temos que $\chi(G) \leq \chi_b(G)$. Como limite superior, observe que qualquer base de ψ define um conjunto de k vértices, cada um com grau pelo menos $k-1$. Assim, se $m(G)$ é o maior inteiro tal que G tem pelo menos $m(G)$ vértices com grau pelo menos $m(G)-1$, então sabemos que ψ não pode ter mais do que $m(G)$ cores. Isto implica que $\chi_b(G) \leq m(G)$. Se $u \in V(G)$ tem grau pelo menos $m(G)-1$, dizemos que u é *denso* e representamos o conjunto de todos os vértices densos de G por $D(G)$. Observe que $m(G)$ é fácil de calcular. Se ordenarmos os vértices de G de forma que $d(v_1) < d(v_2) < \cdots < d(v_n)$, então $m(G) = \max\{k \mid d(v_{n-k+1}) \geq k-1\}$ e podemos encontrar $m(G)$ verificando para quais valores de k temos $d(v_{n-k+1}) \geq k-1$ e retornando o maior deles.

2.4 Estado da arte

O conceito de coloração gulosa foi estudado pela primeira vez na década de 30 [27], mas acredita-se que os primeiros autores a tratarem desse problema em Teoria de Grafos foram Christen e Selkow [13]. Mesmo sendo um problema antigo, apenas em 1997 foi provado por por Goyal e Vishvanathan que decidir se $\Gamma(G) \geq k$ é NP-completo [28]. O problema permanece NP-completo para grafos bipartidos [17] e complementos de grafos bipartidos [29]. Já o problema de decidir se $\Gamma(G) \leq \chi(G) + r$ é coNP-completo [29]. Nesta mesma linha de raciocínio, Asté et al. provaram que decidir se $\Gamma(G) \leq c\chi(G)$ e se $\Gamma(G) \leq c\omega(G)$ também são coNP-completos [30]. Para grafos em geral, Zaker [31] recentemente provou que existe um algoritmo de complexidade $O(n^{2^{k-1}})$ para decidir se $\Gamma(G) \geq k$.

O número de Grundy também foi estudado para algumas classes de grafos. Algoritmos polinomiais existem para determinar o número de Grundy em cografos [32], hipercubos [33], árvores [34] e k-árvores parciais [35, 36].

Alguns limites superiores para o número de Grundy são conhecidos. Se G é o complemento de um grafo bipartido, então $\Gamma(G) \leq 3\omega(G)/2$ [29, 32]. Se G é o complemento de um grafo cordal, então $\Gamma(G) \leq 2\omega(G)+1$ e se G é um grafo split $G = (S \cup K, E)$, então $\Gamma(G) \leq |K|+1$ [32]. Se G é um grafo de intervalos, então $\Gamma(G) \leq 40\omega(G)$ [37]. Este limite foi melhorado recentemente para $8\omega(G)$ [38]. Nikolopoulos e Papadopoulos [39] provaram que $\Gamma(G)$ não pode ser limitado em função de $\omega(G)$ se G é um grafo de permutação. Gyárfás e Lehel [40] foram os primeiros a obter um limite superior para o número de Grundy de grafos livres de P_5. Entretanto, Kierstead et al. [41] melhorou este limite provando que $\Gamma(G) \leq (4^{\omega(G)}-1)/3$ se G é um grafo livre de P_5.

Se G é um grafo com n vértices, a desigualdade de Nordhaus-Gaddum diz que $\chi(G) + \chi(\overline{G}) \leq n+1$ [42]. Zaker [31] encontrou desigualdades similares à de Nordhaus-Gaddum com relação ao número de Grundy para algumas classes de grafos. No caso, ele conjecturou que $\Gamma(G) + \Gamma(\overline{G}) \leq n+2$. Recentemente, foi provado que esta conjectura é válida para grafos bipartidos e grafos com até 8 vértices, porém, a conjectura para grafos em geral é falsa [43].

Um conceito semelhante ao algoritmo de coloração gulosa que é bastante estudado é o *problema de coloração on-line* [26, 32, 44, 45, 46]. Dada uma ordenação $v_1 < v_2 < \cdots < v_n$ dos vértices de G, um algoritmo de coloração on-line atribui cores aos vértices de G em ordem, atribuindo a v_i uma cor que depende apenas do grafo induzido por $\{v_1, \ldots, v_i\}$. Dado um algoritmo on-line $\mathscr{A}$, denotamos por $\chi_{\mathscr{A}}(G)$ o maior número de cores utilizado por $\mathscr{A}$ para colorir G. O *número cromático on-line* de G corresponde ao menor valor $\chi_{\mathscr{A}}(G)$ dentre todos os algoritmos on-line que colorem G.

Os conceitos de b-coloração e número b-cromático foram introduzidos por Irving e Manlove [14]. Eles mostraram que a diferença entre $\chi_b(G)$ e $m(G)$ pode ser arbitrariamente grande para grafos em geral. Em contrapartida, eles mostraram que $\chi_b(G)$ é igual a $m(G)$ ou a $m(G)-1$ quando G é uma árvore e apresentaram um algoritmo que calcula $\chi_b(G)$ para qualquer árvore em tempo polinomial. Além disto, eles provam que decidir se $\chi_b(G) \geq k$ é NP-completo [14]. Kratochvíl, Tuza e Voigt [18] mostram que decidir se $\chi_b(G) = m(G)$ é NP-completo até mesmo se G for um grafo bipartido com exatamente $m(G)$ vértices densos, cada um com grau $m(G)-1$. Um resultado similar é obtido por Havet, Linhares e Sampaio em 2010 [47] para grafos distância-hereditário cordais. Considerando algoritmos aproximativos para o problema, Corteel, Valencia-Pabon, e Vera [48] provam que $\chi_b(G)$ não pode ser aproximado por um fator $120/113-\varepsilon$, para qualquer $\varepsilon > 0$, em tempo polinomial a menos que $P = NP$.

O número b-cromático também foi estudado para algumas classes de grafos. Valores exatos são obtidos para potências de caterpillars completas [49], potências de caminhos [50], potências de árvores d-árias completas [51], grafos de Kneser $K(n,k)$ para alguns valores de n e k [52], hipercubos [53] e grafos cúbicos [54]. Bonomo et al. [55] mostram um algoritmo polinomial para achar o número b-cromático de cografos e grafos P_4-esparsos. Limites inferiores e/ou superiores são obtidos para o número b-cromático são conhecidos para potências de ciclos [50], produto caresiano de grafos completos [56, 57], subgrafos com um vértice removido [58], grafos d-regulares [59, 60, 61], grafos livres de $K_{1,s}$ e grafos bipartidos [62]. Kouider e Zaker obtém limites superiores para o número b-cromático de grafos em geral em função do tamanho da clique máxima e o número de partição em cliques [62].

O número b-cromático do produto cartesiano foi estudado para grafos completos [56, 57] e para estrelas, caminhos e ciclos [53, 63]. Kouider e Mahéo [53] provam que $\chi_b(G \square H) \geq \chi_b(G) + \chi_b(H) - 1$ quando ambos G e H tem b-colorações ótimas cujas bases são conjuntos independentes. Também existem estudos sobre o número b-cromático dos produtos lexicográfico, direto e forte [64].

Para um grafo G, nem sempre existe uma b-coloração de G com k cores para todos os valores de $k \in \{\chi(G), \ldots, \chi_b(G)\}$. Por exemplo, o cubo pode ser b-colorido com 2 ou 4 cores, mas não com 3. Na sua tese, Faik [63] introduz o conceito de b-continuidade de grafos. Um grafo G é *b-contínuo* se G pode ser b-colorido com k cores para todo $k \in \{\chi(G), \ldots, \chi_b(G)\}$. Ele mostra que decidir se um grafo é b-contínuo é NP-completo, até mesmo se G for bipartido e ambos os valores $\chi(G)$ e $\chi_b(G)$ sejam conhecidos. Ele também investiga a b-continuidade de algumas classes de grafos. Em particular, ele prova que grafos cordais são b-contínuos. Outras classes de grafos que são b-contínuas são os grafos de Kneser $K(n,2)$ com $n \geq 17$ [52] e grafos P_4-esparsos [55].

Hoáng e Kouider [65] introduziram e estudaram o conceito de grafos b-perfeitos. Um grafo é b-perfeito se $\chi_b(H) = \chi(H)$ para todo subgrafo induzido H de G. Recentemente, Hoáng, Maffray e Mechebbek [66] caracterizaram todos os grafos b-perfeitos por subgrafos induzidos proibidos.

Noções de b-coloração e o número b-cromático também foram utilizados para resolver problemas de clusterização de dados [67] e no reconhecimento automático de documentos [68].

3 NOVOS LIMITANTES PARA PARA O NÚMERO DE GRUNDY DE PRODUTOS DE GRAFOS

3.1 Introdução

Em 2010, Asté, Havet e Linhares Sales investigaram o número de Grundy de alguns produtos de grafos [69]. Eles mostraram que o número de Grundy do produto lexicográfico de dois grafos é limitado em termos do número de Grundy destes grafos.

Teorema 3.1 ([69]). *Para dois grafos G e H, $\Gamma(G[H]) \leq 2^{\Gamma(G)-1}(\Gamma(H)-1)+\Gamma(G)$.*

Quando G é uma árvore, eles obtiveram o valor exato.

Teorema 3.2 ([69]). *Seja T uma árvore e H um grafo qualquer. Então $\Gamma(T[H]) = \Gamma(T)\Gamma(H)$.*

Eles também mostraram que, em contraste ao produto lexicográfico, não existe nenhum limite superior de $\Gamma(G\Box H)$ como função de $\Gamma(G)$ e $\Gamma(H)$; por exemplo, $\Gamma(K_{p,p})=2$ e $\Gamma(K_{p,p}\Box K_{p,p}) \geq p+1$. No entanto, eles mostraram que $\Gamma(G\Box H)$ é limitado por uma função de $\Delta(G)$ e $\Gamma(H)$.

Teorema 3.3 ([69]). *Para dois grafos G e H, $\Gamma(G\Box H) \leq \Delta(G)\cdot 2^{\Gamma(H)-1}+\Gamma(H)$.*

Naquele trabalho, eles conjecturam que este limite superior está longe do melhor possível.

Conjectura 3.4 ([69]). *Para dois grafos G e H, $\Gamma(G\Box H) \leq (\Delta(G)+1)\Gamma(H)$.*

Essa conjectura generaliza a seguinte conjectura de Balogh, Hartke, Liu e Yu [70].

Conjectura 3.5 ([70]). *Para qualquer grafo H, $\Gamma(K_2\Box H) \leq 2\Gamma(H)$.*

Esta conjectura, por sua vez, foi generalizada como segue:

Conjectura 3.6 (Havet e Zhu). *Se G é um grafo e M é um emparelhamento em G, então $\Gamma(G) \leq 2\Gamma(G\setminus M)$.*

Em [71], Havet, Kaiser e Stehlik provaram a Conjectura 3.4 no caso em que G ou H é uma árvore.

Teorema 3.7 ([71]). *Para um grafo G e uma árvore T, $\Gamma(G\Box T) \leq (\Delta(G)+1)\Gamma(T)$.*

Neste capítulo, aprofundamos a relação entre o número de Grundy do produto direto, produto lexicográfico, produto cartesiano e produto forte de dois grafos e os parâmetros Γ e Δ destes grafos. Primeiro, mostramos que $\Gamma(G\Box H) \leq \Gamma(H[K_{\Delta(G)+1}])$. Em conjunto com os Teoremas 3.1 e 3.2, este resultado implica os Teoremas 3.3 e 3.7 como veremos adiante. Em particular, este método fornece uma demonstração mais curta do Teorema 3.7.

Em seguida, mostramos que $\Gamma(G[K_2]) = \Gamma(G[S_2]\Box K_2)$. Como um corolário deste fato, mostramos um grafo contraexemplo para as Conjecturas 3.4, 3.5 e 3.6: existe um grafo H tal que $\Gamma(H) = 3$ e $\Gamma(K_2\Box H) = 7$. Em conjunto com o Teorema 3.3, isto implica que $\max\{\Gamma(K_2\Box H) \mid \Gamma(H) = 3\} = 7$.

Sobre os produtos direto e forte, respondemos a uma pergunta levantada em [69]. Não existe nenhum limite para $\Gamma(G\times H)$ e $\Gamma(G\boxtimes H)$ como função de $\Gamma(G)$ e $\Gamma(H)$ se $\Gamma(G),\Gamma(H) \geq 3$ (Teorema 3.15). Também mostramos que é impossível limitar $\Gamma(G\times H)$ em função de $\Delta(G)$ e $\Gamma(H)$ quando G é um grafo qualquer não vazio e $\Gamma(H) \geq 5$ (Teorema 3.17). Semelhante ao que acontece com o produto direto, não é possível limitar $\Gamma(G\boxtimes H)$ em função de $\Delta(G)$ e $\Gamma(H)$ quando $\Gamma(H) \geq 5$, a menos que G seja uma união disjunta de grafos completos (Teorema 3.18 e Proposição 3.19).

3.2 Os produtos lexicográfico e cartesiano

O seguinte teorema pode ser utilizado para provar os Teoremas 3.3 e 3.7.

Teorema 3.8. *Para dois grafos G e H, $\Gamma(G\Box H) \leq \Gamma(G[K_{\Delta(H)+1}])$.*

Se utilizarmos o Teorema 3.1, para dois grafos G e H, ficamos com

$$\begin{aligned}\Gamma(G\Box H) &\leq \Gamma(G[K_{\Delta(H)+1}])\\ &\leq 2^{\Gamma(G)-1}(\Gamma(K_{\Delta(H)+1})-1)+\Gamma(G)\\ &= \Delta(H)2^{\Gamma(G)-1}+\Gamma(G)\end{aligned}$$

obtendo uma demonstração alternativa para o Teorema 3.3.

Se utilizarmos o Teorema 3.2, para uma árvore T e um grafo H, ficamos com

$$\begin{aligned}\Gamma(T\Box H) &\leq \Gamma(T[K_{\Delta(H)+1}])\\ &= \Gamma(T)\Gamma(K_{\Delta(H)+1})\\ &= \Gamma(T)(\Delta(H)+1)\end{aligned}$$

obtendo uma demonstração alternativa para o Teorema 3.7. Neste caso, esta demonstração é mais curta do que a demonstração contida em [71].

Asté, Havet e Linhares Sales [69] provaram:

Lema 3.9 ([69])**.** *Para um grafo G e inteiro positivo n, $\Gamma(G[S_n]) = \Gamma(G)$.*

Agora, nós provamos:

Teorema 3.10. *Se G é um grafo, então $\Gamma(G[K_2]) = \Gamma(G[S_2]\Box K_2)$.*

O Teorema 3.10 pode ser generalizado sem muita dificuldade para a sua versão mais geral:

Teorema 3.11. *Se G é um grafo e p um inteiro positivo, então* $\Gamma(G[K_p]) = \Gamma(G[S_p]\Box K_p)$.

O Teorema 3.10 prova que as Conjecturas 3.4, 3.5 e 3.6 são falsas, como mostramos no seguinte resultado:

Corolário 3.12. *Existe um grafo H tal que $\Gamma(H) = 3$ e $\Gamma(K_2\Box H) = 7$.*

De fato, um grafo que construímos no Corolário 3.12 pode ser visto da seguinte forma. Seja G_3 o grafo na Figura 3.1 e $H = G_3[S_2]$. Asté, Havet e Linhares Sales [69] mostram que $\Gamma(G_3) = 3$ e $\Gamma(G_3[K_2]) = 7$. Com isto, o Lema 3.9 implica que $\Gamma(H) = 3$ e o Teorema 3.10 implica que $\Gamma(K_2\Box H) = 7$.

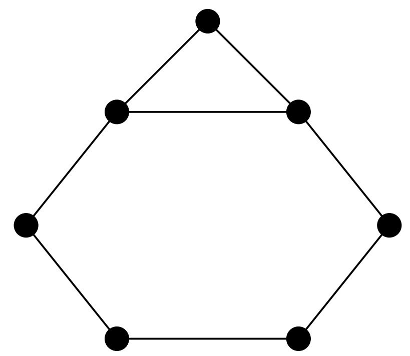


Figura 3.1: Grafo base utilizado na construção dos contraexemplos.

Claramente, o grafo H é um contraexemplo para as Conjecturas 3.4 e 3.5. Note que se v é um vértice de H e a_v, b_v são os dois vértices correspondentes em $K_2\Box H$, então o conjunto $M = \{a_v b_v \mid v \in V(H)\}$ é um emparelhamento em $K_2\Box H$ e $(K_2\Box H)\setminus M$ consiste em duas cópias disjuntas de H sem arestas entre elas; então $\Gamma((K_2\Box H)\setminus M) = 3$. Isto mostra que $K_2\Box H$ é um contraexemplo para a Conjectura 3.6.

O Corolário 3.12 prova que a Conjectura 3.4 não é verdade se $\Gamma(H) = 3$. Por outro lado, mostramos que a Conjectura 3.4 é válida para $\Gamma(H) = 2$.

Proposição 3.13. *Sejam G e H dois grafos. Se $\Gamma(H) = 2$, então* $\Gamma(G\Box H) \le 2(\Delta(G)+1)$.

Na demonstração da Proposição 3.13, utilizamos o fato de que, se H é conexo e $\Gamma(H) = 2$, então H é um grafo bipartido completo [31]. De fato, a demonstração da Proposição 3.13 pode ser generalizada para a Proposição 3.14 que se aplica a grafos multipartidos completos.

Proposição 3.14. *Sejam G um grafo qualquer e H um grafo multipartido completo. Então* $\Gamma(G\Box H) \le \Gamma(H)(\Delta(G)+1)$.

3.3 Os produtos direto e forte

Nesta seção, mostramos que $\Gamma(G \times H)$ e $\Gamma(G \boxtimes H)$ não podem ser limitados por uma função de $\Gamma(G)$ e $\Gamma(H)$ se $\Gamma(G), \Gamma(H) \geq 3$ (Teorema 3.15). Também é natural perguntar se é possível limitar $\Gamma(G \times H)$ ou $\Gamma(G \boxtimes H)$ em função de $\Delta(G)$ e $\Gamma(H)$. Para $\Delta(G) = 1$, temos que $3\lceil \Gamma(H)/2 \rceil - 1 \leq \Gamma(K_2 \times H)$ utilizando um exemplo não trivial apresentado em [30]. Achamos surpreendente que, de fato, não existe limite superior para $\Gamma(K_2 \times H)$ em função de $\Gamma(H)$ se $\Gamma(H) \geq 5$, como mostramos no Teorema 3.17. Uma consequência do Teorema 3.17 é que não é possível limitar $\Gamma(G \times H)$ por uma função de $\Delta(G)$ e $\Gamma(H)$ se $\Delta(G) \geq 1$ e $\Gamma(H) \geq 5$. No Teorema 3.18, mostramos que não existe limite superior para $\Gamma(P_3 \boxtimes H)$ em função de $\Gamma(H)$ se $\Gamma(H) \geq 5$. De fato, o Teorema 3.18 implica que não existe limite superior para $\Gamma(G \boxtimes H)$ como função de $\Delta(G)$ e $\Gamma(H)$ para $\Gamma(H) \geq 5$ a menos que G seja uma união disjunta de grafos completos. Na Proposição 3.19, mostramos que tal limite existe neste caso.

Teorema 3.15. *Para $k \geq 3$, existe um grafo G tal que $\Gamma(G) = 3$, $\Gamma(G \times G) \geq k$ e $\Gamma(G \boxtimes G) \geq k$.*

Seja G o grafo obtido a partir da árvore binomial T_k ao subdividir cada aresta uma vez. Particione os vértices de G em dois conjuntos independentes A e B tal que A contém os vértices originais de T_k e B contém os vértices de subdivisões. Considere qualquer coloração gulosa de G. Qualquer vértice em B tem grau exatamente dois o que implica que recebe uma cor do conjunto $\{1,2,3\}$. Além disto, um vértice em B recebe a cor 3 se, e somente se, seus dois vizinhos recebem as cores 1 e 2. Segue do argumento anterior que nenhum vértice em A pode receber a cor 4. Concluímos que $\Gamma(G) \leq 3$. Como G contém um caminho com quatro vértices para $k \geq 3$, $\Gamma(G) \geq 3$. Então, $\Gamma(G) = 3$. De fato, o grafo G que descrevemos é o grafo utilizado para provar o resultado do Teorema 3.15. A parte mais complicada do Teorema 3.15 consiste em provar que existe um subconjunto de $V(G) \times V(G)$ que induz um subgrafo isomorfo a T_k tanto em $G \times G$ quanto em $G \boxtimes G$. Usamos este subconjunto de vértices para concluir que $\Gamma(G \times G) \geq k$ e $\Gamma(G \boxtimes G) \geq k$.

Para provar os Teoremas 3.17 e 3.18, estudamos o grafo H_k definido como segue. Começamos com uma árvore binomial T_k e particionamos o seu conjunto de vértices em três conjuntos X_1, X_2 e X_3. A raiz de T_k está em X_1. Para qualquer vértice $v \in X_1 \cup X_3$, os filhos de v são colocados em X_2. Para cada vértice $v \in X_2$, os filhos de v são colocados em um conjunto que depende do pai w de v: se $w \in X_1$, então os filhos de v são colocados em X_3; se $w \in X_3$, então os filhos de v são colocados em X_1. Agora, obtemos H_k a partir de T_k adicionando todas as arestas entre X_1 e X_3. A seguir, descrevemos outra forma de visualizar esta partição. Particionamos os vértices de T_k em X_1, X_2 e X_3 dependendo da distância de cada vértice à raiz. Todo vértice cuja distância à raiz for ímpar está em X_2. Todo vértice cuja distância à raiz for da forma $4j$ está em X_1 e e da forma $4j+2$ está em X_3, para algum j inteiro. Assim, os níveis de T_k seguem a ordem $(X_1, X_2, X_3, X_2, X_1, X_2, X_3, \ldots)$. Na Figura 3.2, mostramos o grafo H_5.

Teorema 3.16. *Para $k \geq 1$, $\Gamma(H_k) \leq 5$. Além disto, para $k \geq 9$, $\Gamma(H_k) = 5$.*

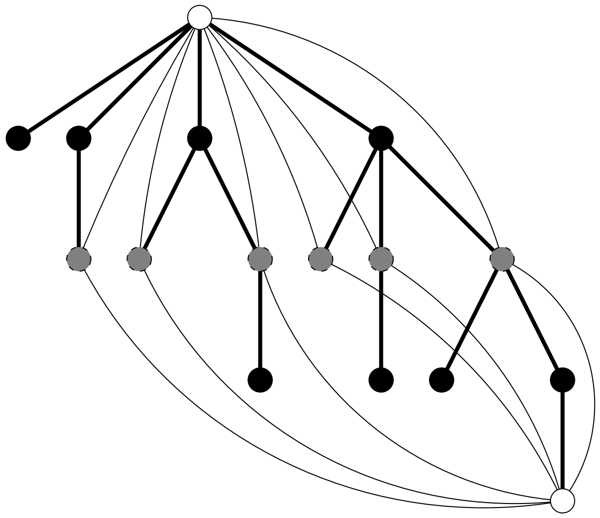


Figura 3.2: Os vértices brancos são os vértices em X_1, os pretos em X_2 e os cinzas em X_3. As arestas grossas são arestas da árvore T_5 e as arestas finas foram adicionadas para formar o grafo H_5.

Note que o raio da árvore binomial T_k é $k-1$. É fácil ver que qualquer árvore com raio no máximo dois tem número de Grundy no máximo três. Esta observação é um corolário do seguinte resultado [25, 26]: *o número de Grundy de uma árvore é igual ao número de Grundy de sua maior árvore binomial induzida*, e do fato de que o raio de uma subárvore de uma árvore T não pode ser maior do que o raio de T.

Mesmo que a demonstração do Teorema 3.16 seja um pouco complicada, a forma em que H_k é construído nos permite ver mais facilmente que $\Gamma(H_k) \leq 6$. Note que qualquer conjunto independente em H_k está contido em $A_1 = X_1 \cup X_2$ ou em $A_2 = X_2 \cup X_3$. Se H_k possui qualquer coloração gulosa com pelo menos sete cores, então pelo menos quatro classes de cor estão contidas em um dos conjuntos A_1 ou A_2, digamos A_j. Isto significa que o subgrafo H^* induzido por A_j em H_k tem número de Grundy pelo menos quatro. No entanto, cada componente de H^* é uma árvore de raio no máximo dois, o que implica que H^* tem número de Grundy no máximo três.

Teorema 3.17. *Se G é um grafo não vazio qualquer e $k \geq 1$, então $\Gamma(G \times H_k) \geq k$.*

Prova. É suficiente provar este teorema quando $G = K_2$. Considere $V(G) = \{v_1, v_2\}$. Agora, afirmamos que $\Gamma(G \times H_k) \geq k$. Para ver que isto é verdade, seja $Y_i = \{v_1\} \times X_i$ para $i \in \{1,3\}$ e $Y_2 = \{v_2\} \times X_2$. Podemos verificar que $Y_1 \cup Y_2 \cup Y_3$ induz uma cópia de T_k em $K_2 \times H_k$, onde o conjunto Y_i toma o papel de X_i na partição de H_k. □

Teorema 3.18. *Se G é um grafo conexo não completo qualquer e $k \geq 1$, então $\Gamma(G \boxtimes H_k) \geq k$.*

Prova. Seja $P_3 = v_1v_2v_3$ um caminho com três vértices. É suficiente provar este teorema quando $G = P_3$, dado que G contém um subgrado induzido isomorfo a P_3. Afirmamos que $\Gamma(G \boxtimes H_k) \geq k$. Para ver que isto é verdade, seja $Y_i = \{v_i\} \times X_i$ para $i \in \{1,2,3\}$. Podemos verificar que $Y_1 \cup Y_2 \cup Y_3$ induz uma cópia de T_k em $P_3 \boxtimes H_k$, onde o conjunto Y_i toma o papel de X_i na partição de H_k. $\square$

Se G é uma união disjunta de grafos completos, então existe um limite superior para $\Gamma(G \boxtimes H)$ em função de $\Gamma(G)$ e $\Gamma(H)$. É suficiente considerar o caso em que $G = K_{m+1}$. Note que $K_{m+1} \boxtimes H = H[K_{m+1}]$. Assim, usando o Teorema 3.1, temos o seguinte resultado.

Proposição 3.19. *Se* $\Gamma(H) = k \geq 2$ *e* $m \geq 1$*, então* $\Gamma(K_{m+1} \boxtimes H) \leq m2^{k-1} + k$*.*

3.4 Comentários

Os resultados deste capítulo estão contidos no artigo que se encontra no Anexo A. Este artigo foi obtido em coautoria com A. Gyárfás, F. Havet, C. Linhares Sales e F. Maffray e foi aceito para publicação no *Journal of Graph Theory* [72].

Na Seção 3.3, mostramos que qualquer limite superior para o número de Grundy de $G \times H$ como função de $\Delta(G)$ e $\Gamma(H)$ é possível apenas para $\Gamma(H) \leq 4$. Talvez, um bom ponto de partida seja verificar se $\Gamma(K_2 \times H)$ é limitado para $\Gamma(H) \leq 4$. Por outro lado, se o grau máximo de ambos os grafos interfere em qualquer limite superior, então conhecemos a desigualdade $\Gamma(G \times H) \leq \Delta(G \times H) + 1 \leq \Delta(G)\Delta(H) + 1$ (que não é um limite muito interessante).

Com respeito ao produto lexicográfico, foi provado em [69] que, se $\Gamma(H) = p$, então para qualquer grafo G, temos que $\Gamma(G[H]) = \Gamma(G[K_p])$. Além disto, como provamos no Teorema 3.11, temos que $\Gamma(G[K_p]) = \Gamma(G[S_p] \square K_p)$. Então, $\Gamma(G[H]) = \Gamma(G[S_p] \square K_p)$. Com isto, o número de Grundy de um produto lexicográfico pode ser visto como um caso particular do número de Grundy do produto cartesiano. Assim, achamos que as perguntas mais interessantes neste domínio são relacionadas ao produto cartesiano. Em particular, mesmo sabendo agora que a Conjectura 3.4 é falsa pelo Corolário 3.12, ainda imaginamos se existe uma constante λ tal que $\Gamma(G \square H) \leq \lambda\,(\Delta(G) + 1)\,\Gamma(H)$ para quaisquer dois grafos G e H. Note que os grafos utilizados na demonstração do Corolário 3.12 satisfazem a seguinte razão $\Gamma(K_2 \square H)/\{(\Delta(K_2) + 1)\,\Gamma(H)\} = 7/6$. Não conseguimos encontrar um grafo com uma razão maior. É verdade que $\Gamma(K_2 \square H) \leq 2c\,\Gamma(H)$ para alguma constante $c \geq 7/6$?

4 B-COLORAÇÃO DE GRAFOS COM POUCOS P_4'S

4.1 Introdução

Muitos problemas NP-Difíceis são resolvidos em tempo polinomial para cografos [23, 73]. Em 2004, o problema de b-coloração foi investigado para cografos em [74], mas este foi inconclusivo em verificar a dificuldade do problema. Já em 2005, o problema foi revisitado considerando a classe dos grafos P_4-esparsos em [65]. Também não se provou se o problema é NP-Difícil para esta classe de grafos.

Finalmente, em 2009, Bonomo, Durán, Maffray, Marenco e Valencia-Pabon [55] provaram que determinar o número b-cromático de grafos P_4-esparsos (e, em particular, cografos) é polinomial. Eles também apresentaram um algoritmo polinomial de programação dinâmica para calcular o número b-cromático para grafos P_4-esparsos. Neste capítulo, nós generalizamos esse resultado para uma vasta quantidade de classes de grafos.

O nosso teorema principal diz que, para qualquer $q > 0$ fixo, temos um algoritmo polinomial para obter o número b-cromático de grafos $(q, q-4)$.

Teorema 4.1. *Para qualquer $q \geq 4$ fixo, existe um algoritmo polinomial para calcular o número b-cromático de grafos $(q, q-4)$.*

Observe que se G tem n vértices, então G tem no máximo $\binom{n}{4}$ P_4's. Assim, G é um grafo $(q, q-4)$ para $q = \binom{n}{4} + 5$. Não podemos esperar que este algoritmo seja polinomial em q, mas o algoritmo que construímos é FPT no menor q tal que G é um grafo $(q, q-4)$.

Na Seção 4.2, mostramos os resultados de [55], incluindo a definição de vetor de dominância e sua relação com as operações de união, junção e aranha. Também descrevemos como calcular o número b-cromático a partir do vetor de dominância. Na Seção 4.3, mostramos os resultados estruturais para grafos $(q, q-4)$ que precisamos para construir nosso algoritmo. Também apresentamos a decomposição primeval, baseada em grafos p-conexos. Na Seção 4.4, mostramos como utilizar a decomposição primeval para provar o Teorema 4.1. Na Seção 4.5, mencionamos as ideias gerais de como provar o Lema 4.6, utilizado para provar o Teorema 4.1, como veremos adiante.

4.2 b-coloração de grafos P_4-esparsos

Seja G um grafo. Dizemos que $v \in V(G)$ é um b-vértice de uma coloração de G se v é adjacente a pelo menos um vértice de cada cor não atribuída a v. O *vetor de dominância* $\mathbf{dom}_G$ é definido como o vetor com índices no conjunto $\{\chi(G), \ldots, |V(G)|\}$ tal que $\mathbf{dom}_G(t)$ é o número máximo de classes de cor distintas que contêm b-vértices dentre todas as colorações de G com t cores [55]. Note que um grafo G admite uma b-coloração com t cores se e somente se $\mathbf{dom}_G[t] = t$. Logo, o número b-cromático $\chi_b(G)$ é o maior número t tal que $\mathbf{dom}_G[t] = t$.

Em [55], Bonomo et al. mostram como obter o número b-cromático de um grafo P_4-esparso em tempo polinomial. Para isto, eles utilizam a decomposição de cografos e grafos P_4-esparsos.

Usando os Teoremas 2.1 e 2.2, foi mostrado em [23] e [73] como calcular o número cromático de grafos P_4-esparsos. Bonomo et al. constroem o vetor de dominância de um cografo G usando o valor de $\chi(G)$, o Teorema 2.1 e o seguinte lema.

Lema 4.2 ([55])**.** *Seja G um grafo obtido pela união ou junção de grafos disjuntos em vértice G_1 e G_2 e $\chi(G) \leq t \leq |V(G)|$.*

Se $G = G_1 \cup G_2$, então $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$ e

$$\mathrm{dom}_G(t) = \min\{t, \mathrm{dom}_{G_1}(t) + \mathrm{dom}_{G_2}(t)\}.$$

Se $G = G_1 \vee G_2$, então $\chi(G) = \chi(G_1) + \chi(G_2)$ e

$$\mathrm{dom}_G(t) = \max_{a \leq j \leq b}\{\mathrm{dom}_{G_1}(j) + \mathrm{dom}_{G_2}(t-j)\}.$$

onde $a = \max\{\chi(G_1), t - |V(G_2)|\}$ e $b = \min\{|V(G_1)|, t - \chi(G_2)\}$.

Para resolver o problema para grafos P_4-esparsos, basta mostrar como calcular o vetor de dominância para grafos aranha.

Lema 4.3 ([55])**.** *Seja G uma aranha com partição (S,C,R), onde $|S| = |C| = k \geq 2$ (se G não tem cabeça, considere $\chi(G[R]) = 0$ e $\mathrm{dom}_{G[R]}(0) = 0$). Então, $\chi(G) = k + \chi(G[R])$ e*

1. *Se G é uma aranha magra, então*

$$\mathrm{dom}_G(i) = \begin{cases} k + \mathrm{dom}_{G[R]}(i-k), & \text{se } \chi(G) \leq i \leq k + |R|, \\ k, & \text{se } i = k + |R| + 1, \\ 0, & \text{se } i > k + |R| + 1. \end{cases}$$

2. *Se G é uma aranha gorda, então*

$$\mathrm{dom}_G(i) = \begin{cases} k + \mathrm{dom}_{G[R]}(i-k), & \text{se } \chi(G) \leq i \leq k + |R|, \\ \min\{k, 4k - 2i + 2|R|\}, & \text{se } k + |R| + 1 \leq i \leq 2k + |R|, \\ 0, & \text{se } i > 2k + |R|. \end{cases}$$

Usando esses lemas, Bonomo et al. provaram o teorema abaixo.

Teorema 4.4 ([55])**.** *O vetor de dominância e o número b-cromático de um cografo ou um grafo P_4-esparso podem ser calculados em tempo $O(n^3)$.*

4.3 Decomposição de grafos $(q, q-4)$

Um grafo é dito (k, ℓ) se nenhum conjunto com até k vértices induz mais que ℓ P_4s distintos. Babel e Olariu [75] estudaram o caso particular de grafos $(q, q-4)$. Dizemos que um grafo é $(q, q-4)$ se qualquer conjunto de até q vértices induz no máximo $q-4$ diferentes P_4's induzidos. Por exemplo, cografos e grafos P_4-esparsos são exatamente os grafos $(4, 0)$ e $(5, 1)$ respectivamente. Os grafos P_4-*estendidos* livres de C_5 são os grafos $(6, 2)$ e os grafos P_4-*leve* são grafos $(7, 3)$ especiais.

Babel e Olariu obtém propriedades para a estrutura de grafos $(q, q-4)$ p-conexos. Utilizando a decomposição primeval, e o Teorema 4.5, sabemos descrever tanto as folhas como os nós internos de qualquer árvore de decomposição primeval de grafos $(q, q-4)$.

Teorema 4.5 ([75])**.** *Se G é um grafo $(q, q-4)$ p-conexo com pelo menos q vértices, então G é um grafo aranha sem cabeça.*

Assim, na decomposição primeval de um grafo $(q, q-4)$, a p-componente H utilizada na aplicação da *operação 3* do Teorema 2.3 é bem conhecida: ou H é uma aranha sem cabeça ou H tem menos de q vértices.

4.4 b-coloração de grafos $(q, q-4)$

Para resolver o problema de b-coloração para um grafo $(q, q-4)$ G, primeiro calculamos a decomposição primeval P_G de G. Aplicando a decomposição de forma recursiva, precisamos apenas achar o vetor de dominância da raiz r de P_G dados os vetores de dominância dos filhos de r. Assim, o próximo passo é calcular o vetor de dominância de G se r é uma folha. Como as folhas da decomposição primeval representam grafos p-conexos, podemos utilizar a caracterização fornecida pelo Teorema 4.5. Qualquer folha da decomposição primeval com pelo menos q vértices é um grafo aranha. Se G é uma aranha, podemos utilizar o Lema 4.3 para calcular $\mathbf{dom}_G$. No caso em que G tem no máximo q vértices, podemos enumerar todas as colorações de G em tempo constante, considerando q fixo.

Agora, precisamos descobrir o vetor de dominância de G considerando que r não é uma folha. Consideramos as possibilidades para os possíveis rótulos de r. Se r for rotulado por *união* ou *junção*, o Lema 4.2 mostra como obter esse vetor. Se r for rotulado pela *operação 3* do Teorema 2.3, temos dois casos a resolver: o grafo p-conexo separável é uma aranha sem cabeça (com $R = \emptyset$) ou é um grafo com menos de q vértices. No primeiro caso, podemos utilizar o Lema 4.3 pois G é uma aranha com cabeça. Resolvemos o segundo caso provando o Lema 4.6.

Seja q um inteiro positivo, H um grafo com menos de q vértices e sejam H_1 e H_2 subgrafos induzidos por uma partição dos vértices de H. Dado um grafo G' com n vértices, seja G o grafo obtido pela aplicação da *operação 3* sobre $(G', H = (H_1, H_2))$, i.e., adicionando todas as arestas entre vértices de G' e H_1. Com relação aos grafos G e G' descritos, podemos enunciar o seguinte lema.

Lema 4.6. *Dados o número cromático $\chi(G')$ e o vetor de dominância* $\mathrm{dom}_{G'}$*, podemos calcular o número cromático $\chi(G)$ em tempo $\Theta(q^{q+2})$ e o vetor de dominância* dom_G *em tempo $\Theta(q^{q+2}n^2)$.*

Em [76], foi provado que o número cromático de um grafo $(q, q-4)$ pode ser calculado em tempo linear utilizando a decomposição primeval. Assim, a dificuldade do Lema 4.6 consiste em calcular o vetor de dominância de G.

Com esse lema, nós temos a prova do nosso resultado principal.

Teorema 4.7. *Se G é um grafo $(q, q-4)$ com n vértices, então $\chi_b(G)$ pode ser calculado em tempo $\Theta(q^{q+2}n^3)$.*

Prova. Calculamos o vetor de dominância de G utilizando a decomposição obtida pelo Teorema 2.3 e os resultados dos Lemas 4.2, 4.3 e 4.6. A partir do vetor de dominância de G, o número b-cromático é o máximo t tal que $\mathrm{dom}_G[t] = t$. □

O Teorema 4.1 é uma consequência direta do Teorema 4.7.

4.5 Discussão sobre a prova do Lema 4.6

Nesta seção, discutimos sobre como provar o Lema 4.6 sem entrar em muitos detalhes. A parte técnica deste lema pode ser encontrada no Anexo B.

Seja G um grafo. Dizemos que $M \subseteq V(G)$ é um módulo de G se todos os vértices de M tem a mesma vizinhança em $V(G) \setminus M$. Seja M um módulo de G, G_M o grafo induzido por M e $N(M)$ a vizinhança dos vértices do módulo. Sejam H, H_1 e H_2 os subgrafos de G induzidos por $V(G) \setminus M$, $N(M)$ e $V(H) \setminus N(M)$, respectivamente. Se H tem menos do que q vértices, o grafo G considerado no Lema 4.6 é obtido ao aplicar a *operação 3* sobre $(G_M, H = (H_1, H_2))$.

Para $\chi(G) \leq t \leq |V(G)|$, dizemos que uma t-coloração ϕ é boa se ϕ tem $\mathrm{dom}_G(t)$ classes de cor com b-vértice. Para calcular $\mathrm{dom}_G(t)$, usamos o Lema 4.8 que mostra que existe uma boa t-coloração de G tal que uma das propriedades a seguir é verdadeira.

1. Todas as cores aparecem em $M \cup V(H_1)$; ou

2. os vértices em M recebem cores distintas.

Dada uma coloração ϕ de G e um subgrafo G' de G, seja $n(\phi)$ o número de cores usadas em ϕ e seja (ϕ, G') a coloração de G restrita ao grafo G'.

Lema 4.8. *Se $\chi(G) \leq t \leq |V(G)|$, então existe uma boa t-coloração de G tal que $n(\phi) = n(\phi, H_1) + n(\phi, G_M)$ ou $n(\phi, G_M) = |V(G_M)|$.*

Aplicando o Lema 4.8, temos quatro casos a analisar:

- (a) Todas as cores aparecem em $M \cup V(H_1)$:
  - (a.1) não tem nenhum b-vértice em H_2;
  - (a.2) tem pelo menos um b-vértice em H_2.
- (b) Os vértices em M tem cores distintas:
  - (b.1) tem cores em M que não estão em H;
  - (b.2) toda cor usada em M aparece em H.

O caso (b.2) é fácil de tratar pois isto implica que $|M| \leq |V(H)|$. Como tratamos o caso em que $|V(H)| < q$, para q fixo, então podemos enumerar todas as t-colorações de G em tempo constante. Para tratar os casos restantes, precisamos de alguma notação adicional.

Seja $\Phi(t)$ o conjunto de todas as t-colorações de H e $\Phi(t,t')$ o subconjunto de $\Phi(t)$ tal que H_1 usa exatamente t' cores. Seja $\phi \in \Phi(t,t')$. Para $H' \subseteq H$, seja $\phi(H')$ o conjunto de cores usadas em H'. Dizemos que um vértice v em H_1 é um *b-vértice parcial* se v é adjacente a pelo menos um vértice de cada cor em $\phi(H_1)$.

Seja $d_1(\phi)$ o número de classes de cor de ϕ com b-vértices parciais em H_1. Seja $d_2(\phi)$ o número de classes de cor de $\phi(H_2) \setminus \phi(H_1)$ com algum b-vértice. Seja $d_3(\phi)$ o número de classes de cor de $\phi(H_1)$ com b-vértice em $V(H_2)$ ou b-vértice parcial em $V(H_1)$. Seja $J \subseteq \phi(H_2) \setminus \phi(H_1)$. Dizemos que um vértice $v \in V(H_1)$ é $\overline{J}$-dominante se v é adjacente a pelo menos um vértice de cada cor em $\phi(H) \setminus J$. Seja $d_4(\phi, J)$ o número de classes de cor de ϕ que tem um b-vértice em $V(H_2)$ ou um vértice $\overline{J}$-dominante em $V(H_1)$. Seja $d_5(\phi, j) = \sup\{d_4(C,J) \mid J \subseteq \phi(H_2) \setminus \phi(H_1), |J| = j\}$.

Seja $\chi(G) \leq t \leq |V|$ e sejam

$$
\begin{aligned}
t_1 &= \max\{t - |V(G_M)|, 0\} \\
t_2 &= \min\{|V(H_1)|, t - \chi(G_M)\} \\
t_3 &= \min\{|V(H)|, t\} \\
t_4 &= \min\{t - |V(G_M)|, |V(H_1)|\} \\
\tau_1(t) &= \sup_{\substack{t_1 \leq t' \leq t_2 \\ t' \leq \hat{t} \leq t_3}} \{\mathrm{dom}_{G_M}(t - t') + d_1(\phi) \mid \phi \in \Phi(\hat{t}, t')\} \\
\tau_2(t) &= \sup_{t_1 \leq t' \leq t_2} \{\min\{t - t', d_2(\phi) + \mathrm{dom}_{G_M}(t - t')\} + d_3(\phi) \mid \phi \in \Phi(t, t')\} \\
\tau_3(t) &= \sup_{\substack{t_1 \leq \hat{t} \leq t_3 \\ 0 \leq t' \leq t_4}} \{d_5(\phi, \hat{t} + |V(G_M)| - t) \mid \phi \in \Phi(\hat{t}, t')\}
\end{aligned}
$$

Se excluirmos o caso (b.2) verificando que $|V(G)| > 2|V(H)|$, temos o Lema 4.1 para auxiliar o cálculo de $\mathrm{dom}_G(t)$.

Lemma 4.1. *Se* $\chi(G) \leq t \leq |V(G)|$ *e* $|V(G)| > 2|V(H)|$*, então*

$$\text{dom}_G(t) = \max\{\tau_1(t), \tau_2(t), \tau_3(t)\}.$$

E com isto podemos provar o Lema 4.6.

Prova do Lema 4.6. Dado que $|V(H)| \leq q$ para um inteiro fixo q, podemos calcular os valores $\tau_1(t)$, $\tau_2(t)$ e $\tau_3(t)$ em tempo linear (uma vez fixados os valores t' e $\hat{t}$, o valor dentro da função sup pode ser obtido em um tempo que depende apenas de q). Se $|V(G)| \leq 2|V(H)| \leq 2q$, então calculamos $\mathsf{dom}_G(t)$ em tempo constante. Se $|V(G)| > 2|V(H)|$, então podemos obter o valor de $\mathsf{dom}_G(t)$ em tempo linear usando o Lema 4.1. Assim, podemos construir o vetor dom_G em tempo $\Theta(n^2)$ para todos os valores de t. □

4.6 Comentários

Os resultados deste capítulo estão contidos no artigo que se encontra no Anexo B. Este artigo foi obtido em coautoria com C. Linhares Sales, A. Maia e R. Sampaio. Um resumo deste artigo foi publicado no *8th French Combinatorial Conference* [77] e um resumo estendido deste artigo foi publicado no *Simpósio Brasileiro de Pesquisa Operacional* [78]. A versão completa deste artigo incluindo este resultado e outros resultados sobre as colorações acíclica, estrela e harmônica está sendo finalizada para submissão na *Discrete Applied Mathematics.*

Neste capítulo, estudamos o número b-cromático de grafos $(q, q-4)$. Apresentamos um algoritmo polinomial para achar o número b-cromático de grafos $(q, q-4)$ para q fixo com complexidade $O(n^3)$. Este algoritmo generaliza o resultado de Bonomo et al. que calcula o número b-cromático de grafos P_4-esparsos com a mesma complexidade. Uma vantagem da técnica utilizada é que, com a mesma complexidade, também podemos responder se existe uma b-coloração de G com k cores, para qualquer k.

As técnicas de Bonomo et al. são adaptáveis para outras classe de grafos. Começamos a investigar o índice b-cromático de grafos catterpillar, onde o índice b-cromático de um grafo G é o maior inteiro k tal que G admite uma b-coloração de arestas com k cores. Esperamos que este estudo inicial e as técnicas utilizadas neste capítulo possam obter um algoritmo polinomial para o índice b-cromático de árvores.

5 B-COLORAÇÃO DE CACTOS

5.1 Introdução

Neste capítulo, generalizamos o resultado de Irving e Manlove [14] para incluir a classe de cactos.

Teorema 5.1 ([14])**.** *Se T é uma árvore, então $\chi_b(T)$ é igual a $m(T)$ ou $m(T)-1$. Além disto, podemos achar uma b-coloração de T com $\chi_b(T)$ cores em tempo polinomial.*

Um grafo é um *cacto* se quaisquer dois ciclos se intersectam em no máximo um vértice. O nosso resultado principal é o seguinte:

Teorema 5.2. *Se G é um cacto com $m(G) \geq 7$, então $\chi_b(G)$ é igual a $m(G)$ ou $m(G)-1$. Além disto, podemos achar uma b-coloração de G com $\chi_b(G)$ cores em tempo polinomial.*

Para provar o Teorema 5.2, primeiro apresentamos uma família de cactos $\mathscr{F}$ tal que $\chi_b(G) = m(G) - 1$ para qualquer $G \in \mathscr{F}$ (Seção 5.2). Para fazer isso, primeiro mostramos que G não pode ser b-colorido com $m(G)$ cores (Seção 5.2) e em seguida mostramos que G pode ser b-colorido com $m(G)-1$ cores (Seção 5.3). Tanto a verificação de que G está em $\mathscr{F}$ como a b-coloração de G podem ser obtidos em tempo polinomial. Em seguida, provamos que, se $G \notin \mathscr{F}$, então G tem um conjunto especial de vértices que chamamos de um conjunto bom de vértices (Seção 5.4); também apresentamos um algoritmo polinomial para encontrar tal conjunto. Finalmente, provamos que se um cacto G tem um conjunto bom de vértices W e $m(G) \geq 7$, então W é a base de uma b-coloração de G com $m(G)$ cores (Seção 5.5).

Estudamos também a seguinte conjectura:

Conjectura 5.3. *Se G é um cacto , então $\chi_b(G)$ é igual a $m(G)$ ou $m(G)-1$.*

Observe que se G é vazio, então $\chi(G) = 1$ e $m(G) = 1$. Se G tem pelo menos uma aresta, então $\chi(G) \geq 2$. Assim, temos que a Conjectura 5.3 é válida para $m(G) \leq 3$. Estudamos também esta conjectura para o caso em que $4 \leq m(G) \leq 6$. Apresentamos uma família de cactos $\mathscr{F}'$ tal que $4 \leq m(G) \leq 6$ e $\chi_b(G) = m(G)-1$ para qualquer $G \in \mathscr{F}'$ (Seção 5.2). Também mostramos que tais grafos são os únicos cactos minimais com $\chi_b(G) < m(G)$ (Seção 5.6).

5.2 Cactos com $\chi_b(G) < m(G)$

Utilizaremos a seguinte propriedade para mostrar que G não admite uma coloração com $m(G)$ cores:

Propriedade 5.4. *Seja ϕ uma b-coloração de G com $m(G)$ cores. Se v é um vértice em qualquer base de G com grau exatamente $m(G)-1$, então todos os vizinhos de v tem cores distintas.*

Seja $W \subseteq D(G)$ com cardinalidade $m(G)$. Se $w, w' \in W$ e $v \in V(G-W)$, então dizemos que w' é uma *ponte* entre v e w se w' tem grau exatamente $m(G)-1$ e é adjacente a ambos v e w. Irving e Manlove definiram um *vértice circulado por* W como um vértice $v \in V(G) \setminus W$ tal que todo $w \in W$ é adjacente a v ou a uma ponte entre v e w. Em seguida, eles provam o seguinte lema importante:

Lema 5.5 ([14])**.** *Seja G um grafo e $W \subseteq D(G)$ um conjunto de cardinalidade $m(G)$ que circula um vértice $v \in V(G) \setminus W$ de grau menor do que $m(G)$. Então, W não é base de uma b-coloração de G com $m(G)$ cores.*

Prova. Se W é base de uma b-coloração ψ com $m(G)$ cores, então seja u o vértice em W com a mesma cor que v. Como ψ é uma coloração própria, u e v não são adjacentes. Assim, temos uma ponte w' entre u e v. Obtemos uma contradição pela Propriedade 5.4. □

Seja G um cacto e W um subconjunto de $m(G)$ vértices densos de G. Dizemos que W *circula o par* $x, y \in V(G)$ se $x, y \notin W$, W não circula nem x nem y e um dos seguintes casos acontece:

E1. Existem $W' \subseteq W$ e $u, v \in W'$ tais que $|W'| = m(G)-1$, $xuyv$ é um ciclo e um dos seguintes é verdadeiro:

   (a) $d(u) = d(v) = m(G)-1$, $N^{W'}(u) \neq \emptyset$, $N^{W'}(v) \neq \emptyset$ e todo $w \in W' \setminus \{u, v\}$ é adjacente a u ou v; ou

   (b) $d(u) = m(G)-1$ e todo $w \in W' \setminus \{u, v\}$ é adjacente a u; ou

   (c) $d(u) = m(G)$, $d(v) = m(G)-1$, $N^{W'}(u) \neq \emptyset$, $N^{W'}(v) \neq \emptyset$ e todo $w \in W \setminus \{u, v\}$ é adjacente a u ou v; ou

   (d) $d(u) = m(G)$ e todo $w \in W \setminus \{u, v\}$ é adjacente a u.

E2. Existem $W' \subseteq W$ e $u, v, w \in W'$ tais que $|W'| \geq m(G)-1$, $xuvyw$ é um ciclo, $d(u) = d(v) = m(G)-1$, todo $w' \in W' \setminus \{u, v, w\}$ é adjacente a w, e um dos seguintes é verdadeiro:

   (a) $W' \subset W$ e $d(w) = m(G)-1$; ou

   (b) $W' = W$ e $d(w) = m(G)$.

Lema 5.6. *Seja G um cacto e $W \subseteq D(G)$ com cardinalidade $m(G)$. Se W circula um par de vértices x, y, então W não é base de uma b-coloração de G com $m(G)$ cores.*

Prova. Por contradição, seja ψ uma coloração de G com base W. Observe que se ocorrerem um dos casos E1a, E1b ou E2a, então todo vértice em W' ou é adjacente ou tem uma ponte tanto para x quanto para y. Pela Propriedade 5.4, ambos x e y recebem a cor do vértice único em $W \setminus W'$. Como x e y tem um vizinho em comum em W com grau $m(G)-1$, então obtemos uma contradição pela Propriedade 5.4.

Assim, podemos considerar que temos um dos casos E1c, E1d ou E2b. Considere $z = u$ se ocorrer o caso E1c ou E1d e $z = w$ se ocorrer o caso E2b. Agora, observe que temos vértices t e t' em $N^W(z)$ (não excluindo a possibilidade que $t = t'$) tal que $\psi(t) = \psi(x)$ e $\psi(t') = \psi(y)$. Como o grau de z é exatamente $m(G)$, obtemos uma contradição pois temos três vértices na sua vizinhança com a mesma cor (caso $\psi(x) = \psi(y)$) ou temos duas cores aparecendo repetidas em sua vizinhança. □

Como sabemos pelo Lema 5.5 e pelo Lema 5.6, se G é um cacto com exatamente $m(G)$ vértices densos e $D(G)$ circula um vértice ou um par de vértices, então $\chi_b(G) < m(G)$. De fato, na demonstração do Teorema 5.2, provamos que se $|D(G)| = m(G)$ e e $\chi_b(G) < m(G)$, então G tem um vértice circulado ou um par de vértices circulados quando $m(G) \geq 7$.

Quando consideramos $4 \leq m(G) \leq 6$, existem grafos com $|D(G)| = m(G)$ e $\chi_b(G) < m(G)$ sem vértices ou pares de vértices circulados. Se H é um grafo na Figura 5.1 ou na Figura 5.2, considere $H' = H - uv$. Como H' é pequeno, podemos verificar que $\chi_b(H') = m(H')$. Na verdade, podemos verificar também que em qualquer b-coloração de H' com $m(H')$ cores, os vértices u e v recebem a mesma cor. Isto implica que $\chi_b(H) < m(H)$.

Dizemos que um cacto G é *anômalo* se $|D(G)| = m(G)$, $G[D(G) \cup N(D(G))]$ contém um grafo isomorfo a algum grafo na Figura 5.1 ou na Figura 5.2 e $d(w) = m(G) - 1$ se w representa um vértice cinza destas figuras. Observe que se G é anômalo, ainda temos que, em qualquer b-coloração de $G - uv$ com $m(G)$ cores, os vértices u e v recebem a mesma cor. Assim, se G é anômalo, então $\chi_b(G) < m(G)$.

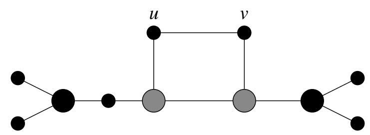


Figura 5.1: Grafo com $|D(G)| = m(G) = 4$ e $\chi_b(G) < m(G)$.

Agora, consideramos cactos com $|D(G)| > m(G)$ tais que $\chi_b(G) < m(G)$. Na demonstração do Teorema 5.2, mostramos que os cactos descritos no Lema 5.7 são os únicos grafos tais que $|D(G)| > m(G)$ e $\chi_b(G) < m(G)$ quando $m(G) \geq 7$.

Lema 5.7. *Se $|D(G)| = m(G) + 1$, existem vértices $u, v \in D(G)$ e $w \notin D(G)$ tais que uvw é um ciclo, $d(u) = d(v) = m(G) - 1$ e todo vértice em $D(G)$ é adjacente a u ou a v, então, $\chi_b(G) < m(G)$.*

Prova. Por contradição, suponha que ψ é uma b-coloração de G com $m(G)$ cores e seja W uma base de ψ. Se $u, v \in W$, então w é um vértice circulado. Caso contrário, sem perda de generalidade, suponha que $u \notin W$. Neste caso, observe que u é um vértice circulado por W. □

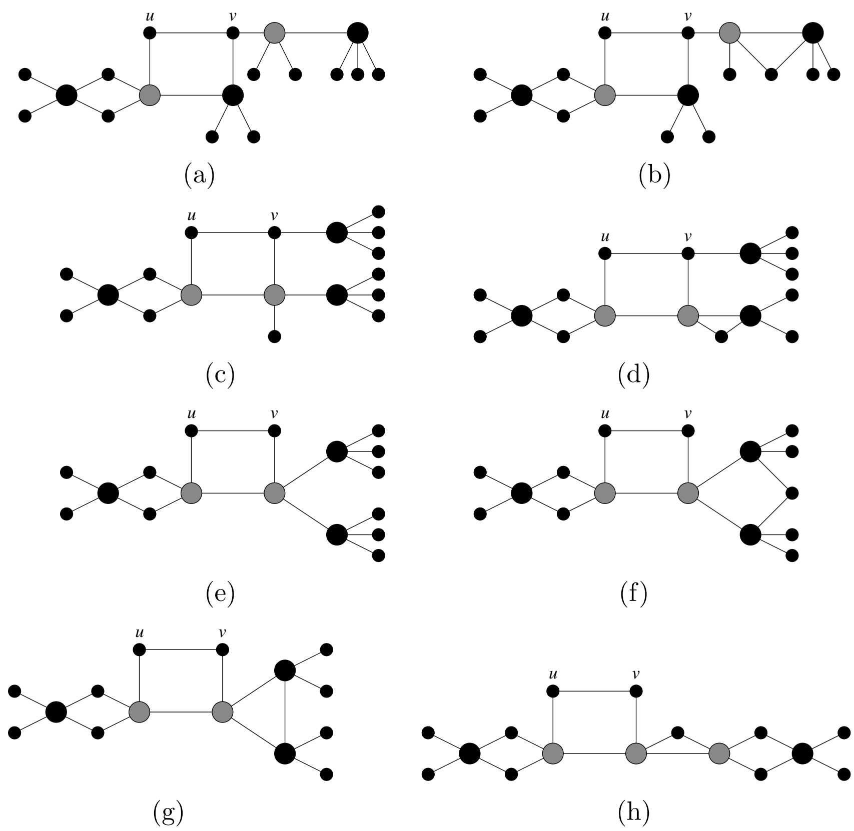


Figura 5.2: Grafos com $|D(G)| = m(G) = 5$ e $\chi_b(G) < m(G)$.

Quando consideramos $4 \leq m(G) \leq 6$, existem outros grafos com $|D(G)| > m(G)$ e $\chi_b(G) < m(G)$. No caso, todos os grafos que conhecemos com estas propriedades tem precisamente $m(G)+1$ vértices densos.

Lema 5.8. *Se H é o grafo da Figura 5.3 ou da Figura 5.4, então $\chi_b(H) < m(H)$.*

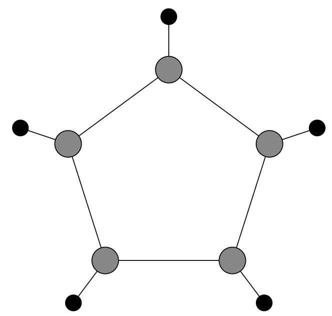


Figura 5.3: Grafo com $|D(G)| = m(G)+1 = 5$ e $\chi_b(G) < m(G)$.

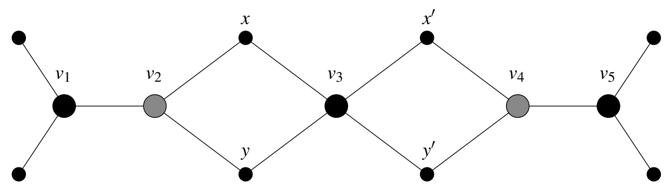


Figura 5.4: Grafo com $|D(G)| = m(G)+1 = 5$ e $\chi_b(G) < m(G)$.

Prova. Seja H o grafo de uma destas figuras. Por contradição, suponha ψ é uma b-coloração de H com $m(H)$ cores e seja W uma base de ϕ. Se H é o grafo da Figura 5.3, então note que o vértice denso que não está em W é circulado por W.

Agora, considere que H é o grafo da Figura 5.4. Seja u o vértice denso que não está em W. Se $u \in \{v_1, v_2\}$, então x' e y' formam um par circulado. Se $u \in \{v_4, v_5\}$, então x e y formam um par circulado. Considere então que $u = v_3$. Neste caso, como $d(v_2) = d(v_4) = m(H) - 1$, então $\psi(\{v_1, v_2\}) = \psi(\{x', y'\})$ e $\psi(\{v_4, v_5\}) = \psi(\{x, y\})$. Assim, v_3 é adjacente a um vértice de cada cor em ψ o que implica que ψ tem pelo menos uma aresta monocromática. $\square$

Dizemos que um cacto G é *problemático* se $|D(G)| = m(G)+1$, $G[D(G) \cup N(D(G))]$ contém um grafo isomorfo ao grafo na Figura 5.3 ou ao grafo na Figura 5.4 e $d(w) = m(G) - 1$ se w representa um vértice cinza destas figuras. Observe que se G é problemático, temos que $\chi_b(G) < m(G)$.

Agora, apresentamos um resumo sobre os grafos descritos nesta seção. Na demonstração do Teorema 5.2, provamos que se $\chi_b(G) < m(G)$, então temos as propriedades da Tabela 5.1 quando $m(G) \geq 7$.

Quando $4 \leq m(G) \leq 6$, além dos grafos descritos na Tabela 5.1, conhecemos os grafos descritos Tabela 5.2 com $\chi_b(G) < m(G)$.

$\lvert D(G)\rvert$	Propriedade
$m(G)$	Vértice circulado ou par circulado
$m(G)+1$	Satisfaz o Lema 5.7

Tabela 5.1: Grafos com $\chi_b(G) < m(G)$ quando $m(G) \geq 7$.

Nome	$\lvert D(G)\rvert$	Referência
Anômalo	$m(G)$	Figura 5.1 e Figura 5.2
Problemático	$m(G)+1$	Figura 5.3 e Figura 5.4

Tabela 5.2: Grafos adicionais com $\chi_b(G) < m(G)$ quando $4 \leq m(G) \leq 6$.

5.3 Colorindo cactos com $\chi_b(G) < m(G)$.

Seja G um cacto com $\chi_b(G) < m(G)$ descrito na Seção 5.2 e seja $W \subseteq D(G)$ com cardinalidade $m(G)-1$. Dizemos que um caminho P entre $u, v \in D(G)$ é um *link de* $D(G)$ se P tem tamanho dois ou três e todo vértice interno de P não está em $D(G)$. Se $u \notin D(G)$ está em um link, dizemos que u é um *vértice link de* $D(G)$. Seja L o conjunto de vértices link de $D(G)$.

Seja W um conjunto de k vértices de G, cada um com grau pelo menos $k-1$. Seja ϕ uma k-coloração de um subconjunto de vértices X de G tal que $W \subseteq X$ e que os vértices de W recebam cores distintas. Para $X' \subseteq V(G)$, seja $\phi(X')$ o conjunto de cores utilizadas em $X \cap X'$. Para cada $w \in W$, seja $M_\phi(w)$ o conjunto de cores $\{1,\ldots,k\} \setminus \phi(N[w])$ e $U_\phi(w)$ o conjunto de vizinhos não coloridos de w. Se não houver ambiguidade, utilizamos apenas $M(w)$ e $U(w)$. Dizemos que ϕ é uma *b-coloração local* se $|M(w)| \leq |U(w)|$, para todo $w \in W$. Dizemos que W é a *base* de ϕ. Note que se $X = V(G)$, então $|M(w)| = |U(w)| = 0$ para todo $w \in W$. Isto implica que os vértices em W são b-vértices e que ϕ é uma b-coloração com k cores.

Lema 5.9. *Se ψ é uma b-coloração local com conjunto candidato para base W que colore os vértices em $D(G) \cup L$, então ψ pode ser utilizada para b-colorir G com $m(G)-1$ cores.*

Prova. Para cada vértice $w \in W$, transforme w em um b-vértice colorindo sua vizinhança. Este passo é possível pois os vértices em L foram coloridos. Em seguida, para cada vértice u não colorido, dê uma cor para u que não aparece em $N(u)$. Este passo é possível pois $u \notin D(G)$ implica que u tem grau no máximo $m(G)-2$. $\square$

No restante desta seção, discutimos como construir uma coloração b-local que colore os vértices em $D(G) \cup L$.

Primeiro, mostramos como b-colorir cactos anômalos ou problemáticos. Observe que para um destes grafos, $D(G) \cup L$ está contido nos vértices das figuras 5.1, 5.2, 5.3 ou 5.4. Assim, uma coloração destes vértices nestas figuras pode ser utilizada para colorir G utilizando o Lema 5.9. Apresentamos estas colorações nas figuras 5.5, 5.6, 5.7

e 5.8. Considere que os vértices cinza destas figuras representam o conjunto candidato para base W. Mostramos apenas a coloração dos vértices em $D(G) \cup L$ e note que estas colorações são válidas sempre que estes grafos estiverem contidos em $G[D(G) \cup N(D(G))]$ quando G é problemático ou anômalo.

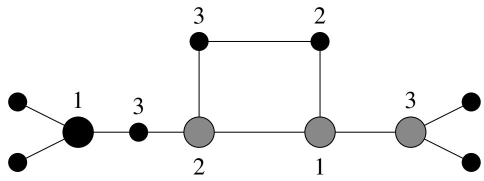


Figura 5.5: b-coloração local de grafos anômalos com $m(G) = 4$.

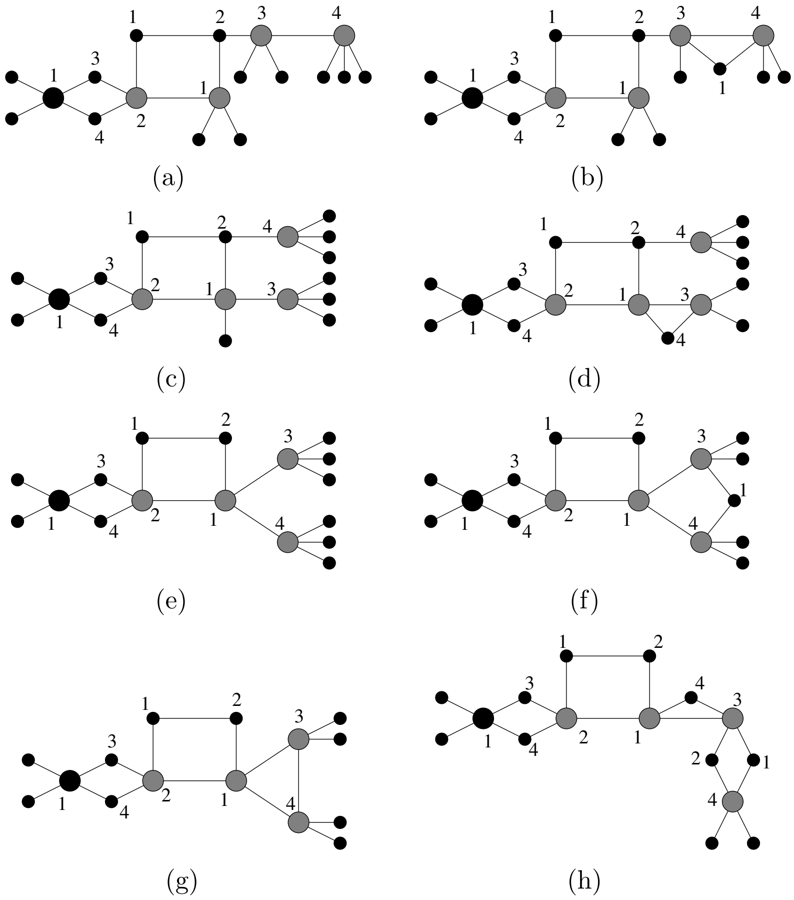


Figura 5.6: b-coloração local de grafos anômalos com $m(G) = 5$.

Agora, consideramos que G é um grafo descrito na Seção 5.2 mas G não é anômalo ou problemático. Neste caso, se $|D(G)| > m(G)$, então G satisfaz as propriedades do Lema 5.7. Se $|D(G)| = m(G)$, então G tem um vértice circulado ou um par de vértices circulados. Para colorir tais grafos, analisamos o número máximo de vértices circulados ou de pares circulados por um conjunto $W \subseteq D(G)$. As seguintes proposições serão úteis:

Proposição 5.10. *Seja $W \subseteq D(G)$ com cardinalidade $m(G)$ e suponha que W circula um vértice $u \in V(G) \setminus W$ com grau menor do que $m(G)$. Então, $|N^W(u)| \geq 2$ e $|W \setminus N(u)| \geq 1$.*

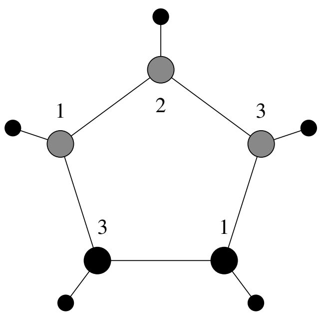


Figura 5.7: b-coloração local de grafos problemáticos.

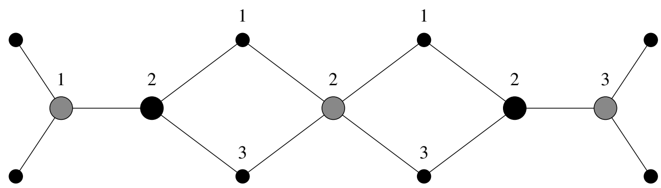


Figura 5.8: b-coloração local de grafos problemáticos.

Prova. Como $d(u) < m(G) = |W|$, existe um vértice $w \in W$ não adjacente a u. Como W circula u, existe um vértice ponte v entre u e w com grau $m(G)-1$. Se v é o único vizinho de u em W, então v é uma ponte entre u e todo vértice em $W \setminus v$. Mas isto é uma contradição pois $d(v) \geq m(G)$. Então, u tem pelo menos dois vizinhos em W. □

Proposição 5.11. *Seja W um conjunto de $m(G)$ vértices densos e seja $u \in V(G) \setminus W$ um vértice circulado por W ou um dos vértices de um par circulado por W. Então, existem pelo menos dois vértices de W adjacentes a u.*

Lema 5.12. *Seja W um subconjunto de $m(G)$ vértices densos. Se W circula dois vértices x e y, então um dos seguintes acontece:*

F1. Existem $u, v \in W$ tais que $xuyv$ é um ciclo, $d(u) = d(v) = m(G)-1$, $N^W(u) \neq \emptyset$, $N^W(v) \neq \emptyset$ e todo $w \in W \setminus \{u, v\}$ é adjacente a u ou v; ou

F2. Existem $u, v, w \in W$ tais que $xuvyw$ é um ciclo, $d(u) = d(v) = d(w) = m(G)-1$ e todo $w' \in W \setminus \{u, v, w\}$ é adjacente a w.

F3. $W = \{v_1, v_2, v_3, v_4\}$, $xv_1v_2yv_3v_4$ é um ciclo em G e $d(v_i) = 3$ para $i \in \{1,2,3,4\}$.

Dados $W \subseteq D(G)$ e $x \in V(G) \setminus W$, utilizamos W_x para representar o conjunto $W \cap (N(x) \cup N(N^W(x)))$. Assim, temos a seguinte observação cuja verificação é trivial.

Observação 5.13. *Se x, y é um par circulado, então $W_x = W_y$ e $|W \setminus W_x| \leq 1$.*

Seja $W \subseteq D(G)$ com cardinalidade $m(G)$. Observe que pela definição de par circulado, W não pode circular tanto um par de vértices como um vértice; também observe que, pelo Lema 5.12, sabemos que W circula no máximo dois vértices distintos. No Lema 5.14, mostramos a situação em que W circula mais de um par de vértices.

Lema 5.14. *Seja W um conjunto qualquer de $m(G)$ vértices densos com $m(G) \geq 4$. Se W circula mais de um par de vértices, então a sua estrutura é como representada na Figura 5.9, onde os vértices grandes representam W e os vértices cinza tem grau $m(G)-1$.*

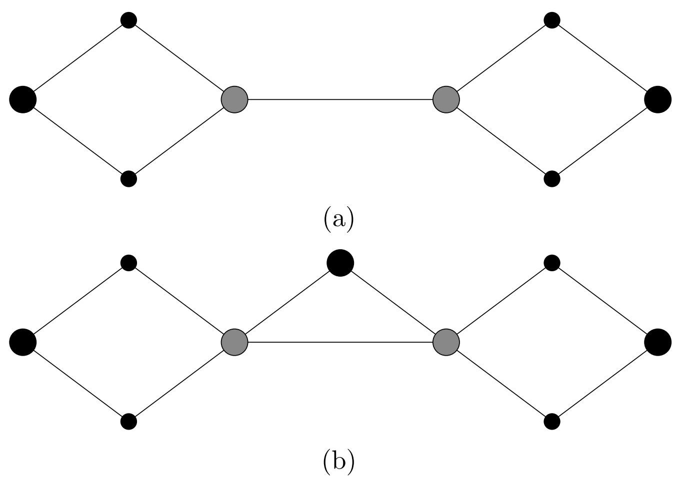


Figura 5.9: Casos em que W circula dois pares de vértices.

Observação 5.15. *Sejam $a,b \in W$. Existem no máximo três links (não necessariamente disjuntos) entre a e b e no máximo dois vizinhos de a e dois vizinhos de b se encontram nestes caminhos(observe a Figura 5.10).*

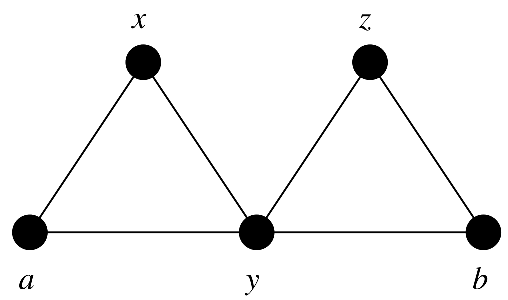


Figura 5.10: Caso com três links entre a e b: ayb, $axyb$ e $ayzb$.

O Lema 5.12 decreve o que acontece quando W circula mais de um vértice o Lema 5.14 descreve o que acontece quando W circula mais de um par de vértices. Na Observação 5.15, descrevemos a relação que podemos ter entre dois vértices de W. Esta descrição nos permite obter uma b-coloração local dos vértices em $D(G) \cup L$. Assim, podemos colorir estes grafos com o Teorema 5.16.

Teorema 5.16. *Se G é um cactus que satisfaz as propriedades do Lema 5.7 ou tal que $|D(G)| = m(G)$ e G tem um vértice circulado ou um par de vértices circulados, então existe uma b-coloração local de G com $m(G)-1$ cores que colore $D(G) \cup L$.*

5.4 Calculando um conjunto bom de vértices

Nesta seção, queremos obter um subconjunto de vértices densos de G para utilizar de base para uma b-coloração de G com $m(G)$ cores. Dizemos que $W \subseteq D(G)$ com cardinalidade $m(G)$ é um *conjunto bom de vértices* se:

- W não circula nenhum vértice ou par de vértices; e
- todo $u \in V \setminus W$ com grau pelo menos $m(G)$ é adjacente a algum vértice em W.

Considere o caso em que $D(G)$ possui exatamente $m(G)$ vértices. Se $D(G)$ circula um vértice ou um par de vértices, então G não tem um conjunto bom de vértices. Se $D(G)$ não circula nenhum vértice e nenhum par de vértices, então $D(G)$ em si é um conjunto bom de vértices. Assim, nos resta analisar a existência de um conjunto bom de vértices em cactos que tem mais do que $m(G)$ vértices densos. O resultado principal desta seção é o seguinte:

Teorema 5.17. *Se G é um cacto com $|D(G)| > m(G)$, então G não tem um conjunto bom de vértices se, e somente se, $D(G) = m(G)+1$ e*

(I) G satisfaz as propriedades do Lema 5.7; ou

(II) G é um grafo problemático.

Suponha que $|D(G)| = m(G)+1$ e (I) ou (II) acontece para $D(G)$. Se G é um grafo problemático contendo o grafo da Figura 5.4 como subgrafo, então observe que o único conjunto de $m(G)$ vértices densos que não circula um par de vértices não é um conjunto bom de vértices. Caso G não seja deste tipo, então observe que qualquer conjunto de $m(G)$ vértices densos circula um vértice. Assim, resta provar que esta condição é suficiente, i.e., se G não tem um conjunto bom de vértices, então $D(G) = m(G)+1$ e (I) ou (II) acontece. Seja $W \subseteq D(G)$ de cardinalidade $m(G)+1$ contendo todos os vértices com grau pelo menos $m(G)$. Na verdade, provamos que, se uma das condições abaixo acontece, então G tem um conjunto bom de vértices:

1. $|D(G)| > m(G)+1$ e (I) ou (II) acontece para W; ou
2. (I) e (II) não acontecem para W e algum $W' \subseteq W$ com cardinalidade $m(G)$ circula dois vértices ou um par de vértices; ou
3. (I) e (II) não acontecem para W e, para todo $W' \subseteq W$ com cardinalidade $m(G)$, W' não circula dois vértices ou um par de vértices.

Seja W' um conjunto de $m(G)$ vértices densos contendo todos os vértices com grau pelo menos $m(G)$. Se G não tem um conjunto bom de vértices, então W' circula pelo menos um vértice ou um par de vértices. Utilizando 2 e 3, temos que (I) ou (II) acontece e utilizando 1, temos que $|D(G)| = m(G)+1$. O teorema principal segue a partir desta argumentação. Agora, apresentamos os lemas que cobrem as situações descritas acima.

Lembrando que $W_x = W \cap (N(x) \cup N(N^W(x)))$, sabemos que se $|W \setminus W_x| \geq 2$, então x não é circulado por W e, pela Observação 5.13, x não faz parte de um par circulado por W. Também observamos que no caso especial em que x é circulado por W, temos que $W_x = W$.

Lema 5.18. *Seja $W \subseteq D(G)$ de cardinalidade $m(G)+1$ contendo todos os vértices com grau pelo menos $m(G)$. Se $|D(G)| > m(G)+1$ e (I) ou (II) acontece para W, então G tem um conjunto bom de vértices.*

Lema 5.19. *Seja $W \subseteq D(G)$ com cardinalidade $m(G)+1$ contendo todos os vértices com grau pelo menos $m(G)$. Se (I) não acontece para W e algum $W' \subset W$ com cardinalidade $m(G)$ circula dois vértices ou um par de vértices, então G tem um conjunto bom de vértices.*

Lema 5.20. *Seja $W \subseteq D(G)$ com cardinalidade $m(G)+1$ contendo todos os vértices com grau pelo menos $m(G)$. Se (I) não acontece para W e todo $W' \subseteq W$ com cardinalidade $m(G)$ não circula dois vértices ou um par de vértices, então G tem um conjunto bom de vértices.*

5.5 Colorindo cactos com um conjunto bom de vértices

O resultado principal desta seção é o seguinte.

Teorema 5.21. *Seja G um cacto com $m(G) \geq 7$ e W um conjunto bom de vértices de G. Então, existe uma b-coloração de G com base W.*

Seja G um cacto com $m(G) \geq 7$. Dado um conjunto bom de vértices W, mostramos como obter uma b-coloração local com conjunto candidato para base W que colore os vértices em $W \cup N(W)$. Depois disto, como $d(v) \leq m(G)-1$, para todo $v \in V(G) \setminus (W \cup N(W))$, podemos colorir os vértices restantes, um a um, dando a v uma cor que não aparece em sua vizinhança.

Denote por G' o subgrafo induzido $G[W \cup N(W)]$ e seja H uma componente conexa de G'. Dizemos que um subconjunto $R \subseteq V(H)$ é um *conjunto compacto* de H se $H[R]$ é conexo e $N(u) \setminus W$ é um subconjunto de R para todo $u \in W \cap R$. Dizemos que R é um *conjunto compacto básico* de H se $R = V(C) \cup \bigcup_{u \in V(C) \cap W} (N(u) \setminus W)$ para algum ciclo C ou se $R = \{u\} \cup (N(u) \setminus W)$, para algum $u \in W \cap V(H)$. Denotamos o conjunto compacto básico definido pelo ciclo C por $[C]$ e o conjunto básico definido por u por $[u]$.

A ideia geral consiste em colorir os vértices de W com cores distintas e depois colorir cada componente conexa H de G separadamente. Para colorir H, utilizamos uma sequência de conjuntos compactos $R_1, \ldots, R_k$ de H onde R_1 é básico, $R_k = V(H)$ e $R_i \subset R_{i+1}$ para $i \in \{1, \ldots, k-1\}$.

Dado um conjunto compacto R de H, dizemos que X é uma R-*aba* se X é o conjunto de vértices de uma componente conexa de $H \setminus R$. Se $u \in N^W(R)$ e $N(u) \setminus (W \cup R) \neq \emptyset$, dizemos que u é um *vértices intermediário* de R.

Antes de explicar como obter esta sequência de conjuntos compactos, mostramos como obter um conjunto compacto básico com uma propriedade conveniente. Para melhor entender a necessidade desta propriedade, considere a sequência de conjuntos compactos $R_1, \ldots, R_k$ de H como mencionado previamente. Suponha que temos uma b-coloração local ψ com conjunto candidato para base W que colore os vértices em $R_i \cup W$. Para estender ψ colorindo os vértices em $R_{i+1} \setminus (R_i \cup W)$, queremos garantir que temos uma quantidade suficiente de vértices de W "distantes" de qualquer R_i-aba X para que possamos utilizar as cores destes vértices distantes para colorir $X \cap (R_{i+1} \setminus (R_i \cup W))$. Agora mais formalmente, queremos que a seguinte propriedade seja válida para todos os conjuntos compactos R na sequência.

(Propriedade Meio) $|X \cap W| \leq \frac{1}{2}|W|$, para toda R-aba X.

A partir de agora, escrevemos que "R satisfaz (PM)" quando o conjunto compacto R satisfaz a Propriedade Meio. Observe que se R é um conjunto compacto que satisfaz (PM) e R' é um conjunto compacto contendo R, então R' também satisfaz (PM). Assim, precisamos apenas garantir que o primeiro conjunto da sequência satisfaz (PM). Provamos a existência de um conjunto compacto básico que satisfaz (PM) no seguinte lema.

Lema 5.22. *Seja H uma componente conexa de G'. Existe um conjunto compacto básico R que satisfaz (PM). Além disto, se $R = [C]$, para algum ciclo C, então não existe um vértice u tal que $R \subseteq (N(u) \setminus W) \cup \{u\}$.*

Agora, queremos construir a sequência de conjuntos compactos começando a partir de um conjunto compacto R satisfazendo o Lema 5.22. Assim, seja R_1 um conjunto compacto básico que satisfaz (PM). Enquanto o conjunto R_i não é igual a $V(H)$, nós obtemos R_{i+1} a partir do conjunto R_i adicionando os vértices em $(N(u) \setminus W) \cup u$, para algum vértice intermediário u de R_i ou $(N(R_i) \cap X) \setminus W$, para alguma R_i-aba X, caso R_i não tenha vértice intermediário.

Pela definição de um conjunto compacto, sabemos que $R \cup [u]$ é um conjunto compacto, para qualquer vértice intermediário u de R. Também sabemos que $R \subset R \cup [u]$. O seguinte lema mostra que se R não tem vértice intermediário, ainda podemos aumentá-lo para obter um conjunto compacto maior.

Lema 5.23. *Seja H uma componente conexa de G' e R um conjunto compacto não vazio de H com $R \neq V(H)$. Se R não tem vértice intermediário, então $R' = R \cup N^X(R)$ é um conjunto compacto para qualquer R-aba X. Além disto, $R \subset R'$.*

Dizemos que uma b-coloração local ψ de G é *boa* para W se ψ é uma b-coloração local com conjunto candidato para base W que colore apenas vértices em G' e tal que, para toda componente conexa H de G', os vértices coloridos em H são precisamente os vértices em $(W \cap V(H)) \cup R$ onde R é vazio ou um conjunto compacto de H que satisfaz (PM). Por simplicidade, como W é o conjunto candidato de ψ e tem que estar colorido, dizemos apenas que ψ colore R.

De agora em diante, seja ψ_0 uma b-coloração local que colore apenas os vértices de W em H e denote o vértice de W colorido com a cor i por w_i, para $i \in \{1, \dots, m(G)\}$. Para colorir H, devemos primeiro apresentar uma b-coloração local boa ψ_1 de G que estende ψ_0 e colore R_1. Em seguida, dados um conjunto compacto R_i e uma b-coloração local boa ψ_i que colore R_i, mostramos como estender ψ_i para uma b-coloração local boa ψ_{i+1} que colore R_{i+1}. Considerem que R_{i+1} é igual a $R_i \cup [u]$ para algum vértice intermediário u se tal vértice existir e que $R_{i+1} = R \cup N^X(R_i)$ para uma R-aba X se R_i não tiver nenhum vértice intermediário. Os seguintes lemas mostram como obter tais colorações.

Lema 5.24. *Seja ψ_0 uma b-coloração local boa que colore apenas os vértices de W em H e seja R um conjunto compacto básico de H. Existe uma b-coloração local boa que estende ψ_0 colorindo R.*

Antes de apresentar o próximo lema, precisamos de algumas definições. Seja R o conjunto de vértices de um ciclo em H ou um conjunto compacto de H e seja ψ uma b-coloração local que colore R. Seja $x \in V(H) \setminus (R \cup W)$. Dizemos que a cor i é *proibida para x* em ψ se existe $w \in N^W(x) \cup \{x\}$ tal que w tem um vizinho colorido com a cor i. Denote o conjunto de cores proibidas para x em ψ por $F_\psi(x)$ (omitimos ψ se não houver ambiguidade). Se R é o conjunto de vértices de um ciclo em H, considere $u \in R \cap W$ e X o conjunto de vértices de todas as componentes conexas de $H - u$ contendo algum vértice em $N(u) \setminus (R \cup W)$. Se R é um conjunto compacto de H, considere u sendo um vértice intermediário de R e X a R-aba contendo u. Recordando que $U(u)$ é o conjunto de vértices não coloridos na vizinhança de u e $M(u)$ é o conjunto de cores que não aparecem na vizinhança de u, observe que $U(u) = N(u) \setminus (R \cup W)$. Seja Q o grafo bipartido $(U(u) \cup M(u), E')$ onde $xc \in E'$ se, e somente se, $c \notin F(x)$. O seguinte lema mostra que existe um emparelhamento em Q que cobre $M(u)$. Este emparelhamento é utilizado para estender ψ.

Lema 5.25. *Sejam R, ψ, u, X e Q como definidos acima. Se $N_Q(c) \neq \emptyset$ para todo $c \in M(u)$ e existem $w_{c_1}, w_{c_2} \in W \setminus X$ tais que $c_1, c_2 \in M(u)$ e $c_1, c_2 \notin F(x)$ para todo $x \in U(u)$, então podemos estender ψ para colorir $R \cup [u]$.*

O lema a seguir estende a b-coloração local boa que colore R no caso em que R é um conjunto compacto sem vértice intermediário.

Lema 5.26. *Seja R um conjunto compacto de H e ψ uma b-coloração local boa que colore R. Se R não tem vértice intermediário e $R \neq V(H)$, então existe uma b-coloração local boa que estende ψ e colore $R' = R \cup N^X(R)$, para alguma R-aba X.*

Prova do Teorema 5.21. Dê uma cor diferente aos vértices de W do conjunto $\{1, \dots, m(G)\}$ para obter uma coloração ψ. Em seguida, para cada componente H de $G' = G[W \cup N(W)]$, estendemos ψ para colorir $V(H)$ da seguinte forma. Seja R o conjunto básico obtido pelo Lema 5.22. Utilizamos o Lema 5.24 para colorir R. Seja ψ^+ a b-coloração local obtida.

Se R não tem vértice intermediário, utilizamos o Lema 5.26 para estender ψ^+ e colorir $R \cup N^X(u)$, para qualquer R-aba X. Tal conjunto é um conjunto compacto pelo Lema 5.23. Assim, suponha que R tem um vértice intermediário u e seja X a R-aba contendo u. Considere o grafo bipartido Q como definido no Lema 5.25. Queremos mostrar como utilizar o Lema 5.25 para estender ψ^+.

1. Existem $w_{c_1}, w_{c_2} \in W \setminus X$ tais que $c_1, c_2 \in M(u)$ e $c_1, c_2 \notin F(x)$ para todo $x \in U(u)$: Como R satisfaz (PM), temos que $|W \setminus X| \geq 4$. Como G é um cacto, sabemos que $|N(u) \cap R| \leq 2$. Assim, seja $\{w_{d_1}, w_{d_2}, w_{d_3}, w_{d_4}\} \subseteq W \setminus X$. Se $|N(u) \cap R| = 2$, então u separa X de R e pelo menos duas cores d_i, d_j estão em $M(u)$ e não estão em $F(x)$, para todo $x \in U(u)$. Caso contrário, existe no máximo uma cor d_i tal que $d_i \notin M(u)$. Como G é um cacto, também temos que no máximo um $x \in U(u)$ pode ter um vizinho em R, x tem no máximo um tal vizinho e o vizinho de x em R não está em W, dado que R é compacto. Assim, no máximo uma cor d_j diferente de d_i aparece na vizinhança de x. Assim, pelo menos duas cores no conjunto $\{d_1, d_2, d_3, d_4\}$ estão em $M(u)$ e estas cores não estão em $F(x')$, para qualquer $x' \in U(u)$.

2. $N_Q(c) \neq \emptyset$, para todo $c \in M(u)$: Considere $c_1, c_2 \in M(u)$ obtidos pela propriedade anterior. Suponha que existe $c \in M(u) \setminus \{c_1, c_2\}$. Como ψ^+ é uma b-coloração local, sabemos que $|U(u)| \geq 3$. Observe que c é cor proibida de no máximo três vértices em $U(u)$. Quando c é proibida para três vértices $x, y, z \in U(u)$, temos que, digamos, x e y são adjacentes a w_c e z é adjacente a um vértice de $R \setminus W$ colorido com a cor c. Mas como $|W \setminus X| \geq 4$, existe um vértice $w_{c'} \in W \setminus (X \cup \{w_{c_1}, w_{c_2}\})$ tal que $c' \in M(u)$. Assim, $|M(u)| \geq 4$ implica que $|U(u)| \geq 4$ e c tem pelo menos um vizinho em Q.

Assim, podemos aplicar o Lema 5.25 para estender ψ^+ colorindo $R \cup [u]$, que é um conjunto compacto de H. Repetimos este procedimento até colorir todos os vértices em H e em seguida para as outras componentes conexas para colorir G'. Depois disto, como W é um conjunto bom de vértices, $d(x) \leq m(G) - 1$ para todo $x \in V(G) \setminus V(G')$ e podemos colorir tais vértices iterativamente pois existe uma cor que não aparece na vizinhança de x. Assim, estendemos a b-coloração local para uma b-coloração de G com $m(G)$ cores. □

5.6 Cactos minimais m-defeituosos

Nesta seção, estamos interessados em estudar a Conjectura 5.3. Para fazer isto, estudamos a estrutura de um cacto G quando $\chi_b(G) < m(G)$. Dizemos que tal grafo é defeituoso. Quando desejamos estudar grafos minimais defeituosos, temos um certo problema. Suponha que G é defeituoso com $m(G)$ suficientemente grande (digamos $m(G) \geq 20$). Acontece que é provável que G contenha um subgrafo H com $m(H)$ bem menor do que $m(G)$. Como um exemplo de tal grafo H, considere o grafo na Figura 5.11.

O problema é que o grafo H não nos diz nada sobre o motivo pelo qual G é defeituoso. Para consertar este problema, definimos grafos m-defeituosos. Dizemos que G é m-defeituoso se $\chi_b(G) < m(G)$ e $m(G) = m$. Dizemos que G é minimal m-defeituoso se G é m-defeituoso e todo subgrafo próprio H de G não é m-defeituoso, i.e., $m(H) < m$ ou $m(H) = m$ e $\chi_b(H) = m(H)$.

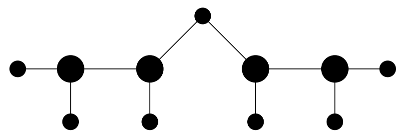


Figura 5.11: Um grafo defeituoso.

Seja G um cacto minimal m-defeituoso. Um dos resultados principais desta seção é relacionado a descrever vértices e arestas desnecessários em G. De forma mais precisa, um resultado consiste no seguinte teorema.

Teorema 5.27. *Se G é um cacto minimal m-defeituoso e $m \geq 4$, então $|D(G)| = m$ e $d(u) = m-1$ para todo $u \in D(G)$.*

Pelo Teorema 5.27, sabemos que os vértices densos em G são incidentes a apenas uma quantidade de arestas necessárias para que estes vértices sejam densos. Também sabemos que temos apenas a quantidade correta de vértices densos para que $m(G) = m$. Utilizando esta ideia, poderíamos esperar que não existam arestas entre vértices que não estão em $D(G)$. Porém, isto é falso como pode-se ver nos grafos anômalos da Figura 5.1 e da Figura 5.2. Já mostramos que se H é um grafo nestas figuras, então H é $m(H)$-defeituoso. Para ver que H é minimal, note que para qualquer $e \in E(H)$, ou temos que $e = uv$ e $\chi_b(H \setminus e) = m(H)$ ou $e \neq uv$ e $m(H \setminus e) < m(H)$.

Mesmo que a ideia de que $V(G) \setminus D(G)$ defina um conjunto independente esteja errada pela existência dos cactos anômalos, não erramos por muito. Para ser mais preciso, o segundo resultado principal desta seção prova que podemos identificar todos os cactos minimais que violam o nosso raciocínio anterior.

Teorema 5.28. *Seja G um cacto minimal m-defeituoso com $m \geq 4$. Se G tem uma aresta entre dois vértices que não estão em $D(G)$, então G é isomorfo a um grafo na Figura 5.1 ou na Figura 5.2 e G tem uma única aresta entre vértices não densos.*

5.7 Comentários

Os resultados deste capítulo se encontram nos Anexos C e D. Esses resultados foram obtido em coautoria com C. Linhares Sales, F. Maffray e A. Silva. Um resumo expandido do artigo contido no Anexo C foi publicado no *LAGOS 2009* [79] e a versão completa deste artigo está sendo finalizada para publicação.

Generalizamos os resultados de Irving e Manlove para cactos com $m(G) \geq 7$. Tal resultado inclui um algoritmo que encontra uma b-coloração ótima de tais cactos. De fato, caracterizamos os cactos que não tem um conjunto bom de vértices e mostramos alguns grafos que, mesmo que tenham um conjunto bom de vértices, não podem ser coloridos com $m(G)$ cores (chamamos tais grafos de anômalos). Mostramos que se G é anômalo ou não tem um conjunto bom de vértices, então $\chi_b(G) = m(G) - 1$. Também mostramos que se G tem um conjunto bom de vértices e $m(G) \geq 7$, então $\chi_b(G) = m(G)$. Conjecturamos que se G tem um conjunto bom de vértices e G não é anômalo com $m(G) \geq 4$, então $\chi_b(G) = m(G)$. Falta provar este resultado para $4 \leq m(G) \leq 6$. Observe que se isto for verdade, então a Conjectura 5.3 é verdadeira. Mostramos resultados iniciais que indicam que os cactos anômalos são os únicos cactos com conjunto bom de vértices tais que $\chi_b(G) \neq m(G)$.

6 CONCLUSÕES

No Capítulo 3, estudamos o número de Grundy dos produtos lexicográfico, cartesiano, direto e forte para grafos gerais. Utilizamos o Teorema 3.8 para relacionar limites superiores do produto cartesiano com o produto lexicográfico. Usando este teorema, apresentamos demonstrações alternativas para os limites conhecidos para o produto cartesiano. Já no Teorema 3.11, mostramos que o número de Grundy do produto lexicográfico pode ser visto como um caso particular do número de Grundy do produto cartesiano. Utilizamos este resultado para achar contra-exemplos para três conjecturas sobre o produto cartesiano e o número de Grundy.

Mostramos, também, a impossibilidade de limitar superiormente o número de Grundy dos produtos direto e forte de G e H por $\Delta(G)$ e $\Gamma(H)$. Qualquer limite superior para o número de Grundy de $G \times H$ ou para $G \boxtimes H$ como função de $\Delta(G)$ e $\Gamma(H)$ é possível apenas para $\Gamma(H) \leq 4$.

No Capítulo 4, estudamos o número b-cromático de grafos $(q, q-4)$. apresentamos um algoritmo polinomial para achar o número b-cromático de grafos $(q, q-4)$ para q fixo com complexidade $O(n^3)$. Este algoritmo generaliza o resultado de Bonomo et al. que calcula o número b-cromático de grafos P_4-esparsos com a mesma complexidade. Uma vantagem da técnica utilizada é que, com a mesma complexidade, também podemos responder se existe uma b-coloração de G com k cores, para qualquer k.

No Capítulo 5, estudamos o número b-cromático de grafos cacto. Generalizamos os resultados de Irving e Manlove para cactos com $m(G) \geq 7$. Se G é um cacto com $m(G) \geq 7$, então $\chi_b(G) \in \{m(G), m(G)-1\}$. Tal resultado inclui um algoritmo que encontra uma b-coloração ótima destes grafos. De fato, caracterizamos os cactos com $\chi_b(G) = m(G) - 1$ e relacionamos este fato a ele não possuir um conjunto bom de vértices. Quando $m(G) < 7$, mostramos os cactos anômalos que, mesmo que tenham um conjunto bom de vértices, não podem ser b-coloridos com $m(G)$ cores. Também mostramos que se G tem um conjunto bom de vértices e $m(G) \geq 7$, então $\chi_b(G) = m(G)$. Conjecturamos que se G tem um conjunto bom de vértices e G não é anômalo com $m(G) \geq 4$, então $\chi_b(G) = m(G)$. Observe que se isto for verdade, então a Conjectura 5.3 é verdadeira. Mostramos resultados iniciais que indicam que os cactos anômalos são os únicos cactos com conjunto bom de vértices tais que $\chi_b(G) \neq m(G)$.

Os resultados apresentados nesta tese foram publicados em forma de artigos científicos em revistas e congressos. Os resultados do Capítulo 3 foram aceitos para publicação no *Journal of Graph Theory* [72]. Os resultados do Capítulo 4 foram publicados*8th French Combinatorial Conference* [77] e no *Simpósio Brasileiro de Pesquisa Operacional* [78]. Os resultados do Capítulo 5 foram publicados no *LAGOS 2009* [79]. Além dos resultados contidos nesta tese, obtivemos outros resultados [80, 81, 82, 83, 84] cujos resumos podem ser encontrados no Anexo E.

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PARTE I

ANEXOS

APÊNDICE A

New bounds on the Grundy number of products of graphs*

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May 23, 2011

Abstract

The Grundy number of a graph G is the largest k such that G has a greedy k-colouring, that is, a colouring with k colours obtained by applying the greedy algorithm according to some ordering of the vertices of G. In this paper, we give new bounds on the Grundy number of the product of two graphs.

1 Introduction

Graphs considered in this paper are undirected, finite and contain neither loops nor multiple edges (unless stated otherwise). The definitions and notation used in this paper are standard and may be found in any textbook on graph theory; see [4] for example. Given two graphs G and H, the *direct product* $G \times H$, the *lexicographic product* $G[H]$, the *Cartesian product* $G \square H$ and the *strong product* $G \boxtimes H$ are the graphs with vertex set $V(G) \times V(H)$ and the following edge sets:

$$\begin{aligned} E(G \times H) &= \{(a,x)(b,y) \mid ab \in E(G) \text{ and } xy \in E(H)\}; \\ E(G[H]) &= \{(a,x)(b,y) \mid \text{either } ab \in E(G), \text{ or } a = b \text{ and } xy \in E(H)\}; \\ E(G \square H) &= \{(a,x)(b,y) \mid \text{either } a = b \text{ and } xy \in E(H), \text{ or } ab \in E(G) \text{ and } x = y\}; \\ E(G \boxtimes H) &= E(G \times H) \cup E(G \square H). \end{aligned}$$

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A *k-colouring* of a graph G is a surjective mapping $\psi : V(G) \to \{1, \dots, k\}$. It is *proper* if for every edge $uv \in E(G)$, $\psi(u) \neq \psi(v)$. A *proper k-colouring* may also be seen as a partition of the vertex set of G into k disjoint non-empty *stable sets* (i.e. sets of pairwise non-adjacent vertices) $C_i = \{v \mid \psi(v) = i\}$ for $1 \leq i \leq k$. For convenience (and with a slight abuse of terminology), by proper k-colouring we mean either the mapping ψ or the partition $\{C_1, \dots, C_k\}$. The elements of $\{1, \dots, k\}$ are called *colours*. A graph is *k-colourable* if it admits a k-colouring. The *chromatic number* $\chi(G)$ is the least k such that G is k-colourable.

Many upper bounds on the chromatic number arise from algorithms that produce colourings. The most basic one is the greedy algorithm. A *greedy colouring* relative to a vertex ordering $v_1 < v_2 < \cdots < v_n$ of $V(G)$ is obtained by colouring the vertices in the order $v_1, \dots, v_n$, assigning to v_i the smallest positive integer not already used on its lower-indexed neighbours. Trivially, a greedy colouring is proper. Denoting by C_i the stable set of vertices coloured i, a greedy colouring has the following property:

For every $i < j$, every vertex in C_j has a neighbour in C_i, $(\star)$

for otherwise the vertex in C_j would have been coloured i or less. Conversely, a colouring satisfying Property $(\star)$ is a greedy colouring relative to any vertex ordering in which the vertices of C_i precede those of C_j whenever $i < j$. The *Grundy number* $\Gamma(G)$ is the largest k such that G has a greedy k-colouring.

Let $\Delta(G)$ denote the maximum degree in a graph G. Let K_n denote the complete graph on n vertices and $K_{p,q}$ denote the complete bipartite graph with parts of size p and q. Let S_n denote the edgeless graph on n vertices.

In [1], Asté, Havet and Linhares Sales investigated the Grundy number of several types of graph products. They showed that the Grundy number of the lexicographic product of two graphs is bounded in terms of the Grundy numbers of these graphs.

Theorem 1 ([1]). *For any two graphs G and H, $\Gamma(G[H]) \leq 2^{\Gamma(G)-1}(\Gamma(H) - 1) + \Gamma(G)$.*

Moreover, when the graph G is a tree, they obtained an exact value.

Theorem 2 ([1]). *Let T be a tree and H be any graph. Then $\Gamma(T[H]) = \Gamma(T)\Gamma(H)$.*

They also showed that, in contrast with the lexicographic product, there is no upper bound of $\Gamma(G \square H)$ as a function of $\Gamma(G)$ and $\Gamma(H)$; for example, $\Gamma(K_{p,p}) = 2$ and $\Gamma(K_{p,p} \square K_{p,p}) \geq p + 1$. Nevertheless, they showed that $\Gamma(G \square H)$ is bounded by a function of $\Delta(G)$ and $\Gamma(H)$.

Theorem 3 ([1]). *For any two graphs G and H, $\Gamma(G \square H) \leq \Delta(G) \cdot 2^{\Gamma(H)-1} + \Gamma(H)$.*

However, they conjectured that this upper bound is far from being tight.

Conjecture 4 ([1]). *For any two graphs G and H, $\Gamma(G \square H) \leq (\Delta(G) + 1)\, \Gamma(H)$.*

This conjecture generalises the following conjecture of Balogh, Hartke, Liu and Yu [3].

Conjecture 5 ([3])**.** *For any graph H, $\Gamma(K_2 \Box H) \leq 2\Gamma(H)$.*

Here is another conjecture that would imply the preceding one.

Conjecture 6 (Havet and Zhu)**.** *If G is any graph and M is a matching in G, then $\Gamma(G) \leq 2\Gamma(G \setminus M)$.*

In [7], Havet, Kaiser and Stehlik proved Conjecture 4 in the case when one of G, H is a tree.

Theorem 7 ([7])**.** *For any graph G and tree T, $\Gamma(G \Box T) \leq (\Delta(G) + 1)\,\Gamma(T)$.*

Here we investigate further the relation between the Grundy number of the direct product, lexicographic product or Cartesian product of two graphs and the invariants Γ and Δ of the two graphs. We first show that $\Gamma(G \Box H) \leq \Gamma(H[K_{\Delta(G)+1}])$. Together, with Theorem 1 and 2, this implies Theorems 3 and 7 respectively. In particular, we obtain a shorter proof of Theorem 7.

We then show that $\Gamma(G[K_2]) = \Gamma(G[S_2] \Box K_2)$. As a corollary, we give an example of a graph that disproves Conjectures 4, 5 and 6: there is a graph H such that $\Gamma(H) = 3$ and $\Gamma(K_2 \Box H) = 7$. Together with Theorem 3 this yields $\max\{\Gamma(K_2 \Box H) \mid \Gamma(H) = 3\} = 7$.

Regarding the direct and strong product, we answer a question raised as the last sentence in [1]. There cannot be any bound on $\Gamma(G \times H)$ and $\Gamma(G \boxtimes H)$ as a function of $\Gamma(G)$ and $\Gamma(H)$ if $\Gamma(G)$ and $\Gamma(H)$ are both greater than or equal to 3 (Theorem 15). It is also impossible to bound $\Gamma(G \times H)$ in terms of $\Delta(G)$ and $\Gamma(H)$ when G is any graph with at least one edge and $\Gamma(H) \geq 5$ (Theorems 17). Similarly, it is impossible to bound $\Gamma(G \boxtimes H)$ in terms of $\Delta(G)$ and $\Gamma(H)$ when $\Gamma(H) \geq 5$ unless G is the disjoint union of complete graphs (Theorem 18 and Proposition 19).

2 The Cartesian and lexicographic products

2.1 Common proof of Theorems 3 and 7

Theorem 8. *For any two graphs G and H, $\Gamma(G \Box H) \leq \Gamma(G[K_{\Delta(H)+1}])$.*

Proof. We shall prove that if $G \Box H$ has a greedy q-colouring for some integer q, then so does $G[K_{\Delta(H)+1}]$. Hence consider a greedy q-colouring φ of $G \Box H$.

Let $x_1, x_2, \ldots, x_n$ be an ordering of the vertices of G such that $\varphi(x_1, y) \leq \varphi(x_2, y) \leq \cdots \leq \varphi(x_n, y)$. Let $z_0, \ldots, z_{\Delta(H)}$ be the vertices of $K_{\Delta(H)+1}$. So every vertex of $G[K_{\Delta(H)+1}]$ is a pair (x_i, z_j) for some $i \in \{1, \ldots, n\}$ and $j \in \{0, 1, \ldots, \Delta(H)\}$. Let (x, y) be a vertex of $G \Box H$ with colour q.

For every i in $\{1, \ldots, n\}$, we assign colour $\varphi(x_i, y)$ to vertex (x_i, z_0) of $G[K_{\Delta(H)+1}]$. Then for $i = 1$ to n, we do the following. Let L_i be the set of all colours ℓ, with $\ell < \varphi(x_i, y)$, that have not been assigned to any neighbour of (x_i, z_0) in $G[K_{\Delta(H)+1}]$. Since φ is a greedy colouring and colour $\varphi(x_j, y)$ is assigned to (x_j, z_0) for each j, L_i is a subset of $\{\varphi(x_i, u) \mid u \in N(y)\}$. Therefore $|L_i| \leq \Delta(H)$. Hence we can assign all the colours of L_i to distinct vertices in $\{(x_i, z_j) \mid 1 \leq j \leq \Delta(H)\}$.

Let us show that the obtained partial q-colouring of $G[K_{\Delta(H)+1}]$ is a greedy colouring. It is proper since colours already assigned to neighbours of (x_i, z_0) are not in L_i. In L_i we add every colour $\ell < \varphi(x_i, z_0)$ such that (x_i, z_0) had no neighbour coloured ℓ before Step i. Hence, after Step i, vertex (x_i, z_0) has a neighbour of each colour less than $\varphi(x_i, y)$. Now every coloured vertex (x_i, z) has a colour ℓ less than $\varphi(x_i, y)$. But, by the definition of the lexicographic product, all neighbours of (x_i, z_0), except (x_i, z) itself, are neighbours of (x_i, z). Hence (x_i, z) has a neighbour of each colour less than ℓ. So the colouring is greedy. $\square$

2.2 Disproof of Conjecture 4

Asté, Havet and Linhares Sales [1] proved the following:

Lemma 9 ([1])**.** *For any graph G and any integer n, $\Gamma(G[S_n]) = \Gamma(G)$.*

Now we prove:

Theorem 10. *Let G be a graph. Then $\Gamma(G[K_2]) = \Gamma(G[S_2]\Box K_2)$.*

Proof. Let us show that the left hand side is at most the right hand side. Consider a greedy colouring φ of $G[K_2]$. Every vertex v of G corresponds to two adjacent vertices of $G[K_2]$. Let us denote by $\varphi_1(v)$ and $\varphi_2(v)$ the two distinct colours assigned by φ to these vertices. In the graph $G[S_2]\Box K_2$, every vertex v corresponds to four vertices a_v, b_v, a'_v and b'_v inducing two edges $a_v b_v$ and $a'_v b'_v$, and so that if uv is any edge of G, then $G[S_2]\Box K_2$ has all edges between $\{a_u, a'_u\}$ and $\{a_v, a'_v\}$ and all edges between $\{b_u, b'_u\}$ and $\{b_v, b'_v\}$. Assign colour $\varphi_1(v)$ to a_v and b'_v and colour $\varphi_2(v)$ to b_v and a'_v. Doing this for every vertex, it is easy to check that we obtain a greedy colouring of $G[S_2]\Box K_2$. Hence $\Gamma(G[K_2]) \leq \Gamma(G[S_2]\Box K_2)$.

Let us now show that the right hand side is at most the left hand side. By Theorem 8, we have $\Gamma(G[S_2]\Box K_2) \leq \Gamma(G[S_2][K_2])$. We claim that $\Gamma(G[S_2][K_2]) \leq \Gamma(G[K_2])$. To see this, consider any greedy colouring φ of $G[S_2][K_2]$ with q colours. In $G[S_2][K_2]$, every vertex v of G corresponds to four vertices a_v, b_v, c_v, d_v with two edges $a_v b_v, c_v d_v$, and for every edge uv of G, there are all edges between $\{a_u, b_u, c_u, d_u\}$ and $\{a_v, b_v, c_v, d_v\}$. Suppose that φ assigns at least three different colours in $\{a_v, b_v, c_v, d_v\}$ for some v, say $\varphi(a_v) = i$, $\varphi(b_v) = j$, $\varphi(c_v) = k$, where, up to symmetry, $i < j$ and $k \notin \{i, j\}$. Note that b_v has no neighbour of colour k, because its neighbours are either a_v or adjacent to c_v. So $j < k$. At least one colour $h \in \{i, j\}$ is not the colour of d_v, so c_v has no neighbour of colour h, a contradiction. So φ uses exactly two colours in $\{a_v, b_v, c_v, d_v\}$ for every vertex v of G. It follows that the restriction of φ on the subgraph of $G[S_2][K_2]$ induced by $\{a_v, b_v \mid v \in V(G)\}$, which is isomorphic to $G[K_2]$, is a greedy colouring with q colours. So the claim that $\Gamma(G[S_2]\Box K_2) \leq \Gamma(G[K_2])$ is established. This completes the proof. $\square$

Remark 11. Theorem 10 can be generalised in a straightforward manner to the following result: *Let G be any graph and p be any integer. Then* $\Gamma(G[K_p]) = \Gamma(G[S_p]\square K_p)$.

Theorem 10 implies that Conjectures 4, 5 and 6 do not hold, as follows.

Corollary 12. *There is a graph H such that $\Gamma(H) = 3$ and $\Gamma(K_2\square H) = 7$.*

Proof. Let G_3 be the graph that consists of a cycle of length 6 plus one vertex g adjacent to a vertex a of the cycle and one vertex h adjacent to another vertex b of the cycle, where a and b are adjacent. Let $H = G_3[S_2]$. Asté, Havet and Linhares Sales [1] showed that $\Gamma(G_3) = 3$ and $\Gamma(G_3[K_2]) = 7$. Hence, Lemma 9 yields $\Gamma(H) = 3$ and Theorem 10 yields $\Gamma(K_2\square H) = 7$. This proves the corollary.

Alternately, let G_3' be the graph obtained from G_3 by identifying the two vertices g and h (i.e., replacing them by one vertex adjacent to a and b), and let $H' = G_3'[S_2]$. Then one can also check that $\Gamma(H') = 3$ and $\Gamma(K_2\square H') = 7$. □

Clearly, the two graphs H and H' mentioned in the preceding proof are counterexamples to Conjectures 4 and 5. Note also that if v is any vertex of H and a_v, b_v are the corresponding two vertices in $K_2\square H$, then the set $M = \{a_v b_v \mid v \in V(H)\}$ is a matching in $K_2\square H$, and $(K_2\square H)\setminus M$ consists of two disjoint copies of H with no edge between them; so $\Gamma((K_2\square H)\setminus M) = 3$. This shows that $K_2\square H$ is a counterexample to Conjecture 6. The same holds for $K_2\square H'$.

Corollary 12 shows that Conjecture 4 does not hold if $\Gamma(H) = 3$. On the other hand, we now show that Conjecture 4 holds if $\Gamma(H) = 2$.

Proposition 13. *Let G and H be two graphs. If $\Gamma(H) = 2$ then $\Gamma(G\square H) \le 2(\Delta(G) + 1)$.*

Proof. If H is not connected, and has components $H_1, \dots, H_p$ ($p \ge 2$), then $G\square H$ is the disjoint union of $G\square H_1, \dots, G\square H_p$, and it suffices to prove the Proposition for each graph $G\square H_i$. Therefore we may assume that H is connected. If $\Gamma(H) = 2$ then H is a complete bipartite graph [10]. Let (A, B) be its bipartition. For every vertex $v \in V(G)$, define $A_v = \{(v, a) \mid a \in A\}$ and $B_v = \{(v, b) \mid b \in B\}$, so A_v and B_v are the two sides of the copy of H indexed by v in $G\square H$. Let φ be a greedy colouring of $G\square H$. We claim that:

For any $v \in V(G)$, $|\varphi(A_v)| \le \Delta(G) + 1$ and $|\varphi(B_v)| \le \Delta(G) + 1$.

Assume for a contradiction, and up to symmetry, that $|\varphi(A_v)| \ge \Delta(G) + 2$. Let α be the largest colour of $\varphi(A_v)$ and let $x = (v, a)$ be a vertex coloured α. The neighbourhood of x in $G\square H$ is $B_v \cup \{(w, a) \mid w \in N_G(v)\}$. But the colours of $\varphi(A_v)$ do not appear on B_v because it is complete to A_v, and $|\{(w, a) \mid w \in N_G(v)\}| = d_G(v) \le \Delta(G)$. Hence at most $\Delta(G)$ colours of $\varphi(A_v)$ may appear on the neighbourhood of x, and so at least one colour of $\varphi(A_v) \setminus \{\alpha\}$ does not. This contradicts the fact that φ is a greedy colouring and proves the claim.

Let $y = (v, b)$ be a vertex such that $\varphi(y)$ is maximum. Without loss of generality, we may assume that $b \in B$. At most $2\Delta(G) + 1$ colours appear in the neighbourhood of y: at most $\Delta(G) + 1$ on A_v according to the claim, and at most one more for each of its neighbours not in B_v, whose number is $d_G(y) \leq \Delta(G)$. Hence $\varphi(y) \leq 2\Delta(G) + 2$. $\square$

Remark 14. Proposition 13 can easily be generalised to complete multipartite graphs in a straightforward manner to obtain the following result: *if H is a complete multipartite graph, then* $\Gamma(G \square H) \leq (\Delta(G) + 1)\,\Gamma(H)$.

3 The direct and strong products

Here we show that $\Gamma(G \times H)$ and $\Gamma(G \boxtimes H)$ cannot be bounded by a function of $\Gamma(G)$ and $\Gamma(H)$ if $\Gamma(G), \Gamma(H) \geq 3$ (Theorem 15). It is also a natural question to bound $\Gamma(G \times H)$ or $\Gamma(G \boxtimes H)$ in terms of $\Delta(G)$ and $\Gamma(H)$. For $\Delta(G) = 1$, a non-trivial construction of [2] shows that $3\lceil \Gamma(H)/2 \rceil - 1 \leq \Gamma(K_2 \times H)$. Somewhat surprisingly, we show in Theorem 17 that there is no upper bound on $\Gamma(K_2 \times H)$ in terms of $\Gamma(H)$ if $\Gamma(H) \geq 5$. Moreover, we show in Theorem 18 that there is no upper bound on $\Gamma(P_3 \boxtimes H)$ in terms of $\Gamma(H)$ if $\Gamma(H) \geq 5$. In fact, Theorem 18 implies that there is no upper bound on $\Gamma(G \boxtimes H)$ as a function $\Delta(G)$ and $\Gamma(H)$ for $\Gamma(H) \geq 5$ unless G is the disjoint union of complete graphs. In Proposition 19, we show that there is an upper bound in such a case.

Let us first recall some definitions. The *binomial tree* is the graph T_k defined recursively as follows. For $k = 1$, T_1 is the one-vertex graph. For $k \geq 2$, T_k is obtained from T_{k-1} by adding, for each vertex v of T_{k-1}, one vertex v' with an edge vv'. It is easy to see that, for $k \geq 2$, T_k has two adjacent vertices r, s of degree $k - 1$ and the other vertices have degree at most $k - 2$, and the two components of $T_k \setminus rs$ are both isomorphic to T_{k-1}. We view T_k as rooted at vertex r. We have $\Gamma(T_k) = k$. More precisely, T_k has a greedy colouring ψ where each vertex $v \notin \{r, s\}$ has colour equal to its degree, and s, r have colour $k - 1$ and k respectively. Note that for each vertex v and colour $i < \psi(v)$, v has a unique neighbour of colour i.

The *radius* of a graph G is the smallest integer t for which there exists a vertex a of G such that every vertex of G is at distance at most t from a. Note that the radius of T_k is $k-1$. It is easy to see that every tree with radius at most 2 has Grundy number at most 3. This is also a corollary of the following result from [5, 6]: *the Grundy number of a tree is equal to the Grundy number of its largest binomial subtree*, and of the fact that the radius of a subtree of a tree T is not larger than the radius of T.

Theorem 15. *For every $k \geq 3$, there is a graph G such that $\Gamma(G) = 3$ and $\Gamma(G \times G) \geq k$ and $\Gamma(G \boxtimes G) \geq k$.*

Proof. Let G be the graph obtained from T_k by subdividing every edge once. Partition the vertex set of G into two stable sets A and B such that A contains the original vertices of T_k and B contains the subdivision vertices. Consider any greedy colouring of G. Every vertex in B has degree 2 and consequently receives a colour from the set $\{1, 2, 3\}$. Moreover, a vertex in B receives colour 3 if and only if its two neighbours have received colours 1 and 2 respectively. It follows that no vertex of A can receive colour 4 or more. This implies that $\Gamma(G) \leq 3$. Since G contains a four-vertex path, $\Gamma(G) \geq 3$. Thus $\Gamma(G) = 3$. To complete the proof of the theorem, let us show that $G \times G$ and $G \boxtimes G$ have a common induced subgraph H_k isomorphic to T_k. This implies $\Gamma(G \times G) \geq k$ and $\Gamma(G \boxtimes G) \geq k$.

Let the root r of T_k become the root of G. Since G is viewed as a rooted tree, every vertex in B has one parent and one child. Consider the greedy colouring ψ of T_k with k colours as defined above, such that the root r has colour k and the second vertex s of degree $k-1$ has colour $k-1$. For $i \in \{1, \dots, k\}$, let A_i be the set of vertices in A that receive colour $(k+1)-i$. So $A_1 = \{r\}$ and $A_2 = \{s\}$. For each $i \in \{2, \dots, k\}$, let B_i be the set of vertices in B whose child is in A_i. We say that a vertex v in $A_i \cup B_i$ has *label* i and denote by ℓ_v the label of v. Let q be the parent of s (i.e., q is the common neighbour of r and s). Let $d(x, y)$ denote the distance between any two vertices x and y in G. We prove by induction on $i \in \{2, \dots, k\}$ that $G \times G$ and $G \boxtimes G$ have an induced subgraph H_i such that:

(1) H_i is isomorphic to T_i and contains vertex (r, q).
(2) Every vertex of H_i is of the form (a, b) or (b, a), with $a \in A$ and $b \in B$; moreover, $\ell_a < \ell_b \leq i$, vertices a, b lie in distinct components of $G \setminus rq$, and $d(a, r) = d(b, q)$.

For $i = 2$, the induced subgraph H_2 with vertices (r, q) and (q, r) and an edge between them is the desired copy of T_2. Now let $i \geq 3$. By the induction hypothesis, there exists a common induced subgraph H_{i-1} of $G \times G$ and $G \boxtimes G$ that satisfies (1) and (2). Let z be any vertex of H_{i-1}, and let $a \in A$ and $b \in B$ be such that z is equal to (a, b) or (b, a). Let u be the unique child of b in G. By the definition of the labels we have $\ell_u = \ell_b$. By property (2), we have $\ell_a \leq i-1$, so (in T_k, and since ψ is a greedy colouring) a has a neighbour of colour $(k+1)-i$, and (in G) a has a neighbour $v \in B$ with label i. Clearly, u and v lie in distinct components of $G \setminus rq$ since a and b do. Now, either (v, u) or (u, v) is a neighbour of (a, b) in $G \times G$ and we call this neighbour the *leaf* of z, and z is called the *support* of its leaf. Note that any leaf-support edge is also an edge in $G \boxtimes G$ as $E(G \times G) \subseteq E(G \boxtimes G)$. Since v has label i, the leaf of z is not a vertex in H_{i-1}. Since $\ell_u = \ell_b \leq i-1$ and $\ell_v = i$, we have $\ell_u < \ell_v \leq i$. Since u is a child of b and v is a child of a, we have $d(u, r) = d(v, q)$. (More precisely: if a lies in the component G_r of $G \setminus rq$ that contains r and b lies in the other component G_q, then $d(u, r) = d(b, q) + 2$ and $d(v, q) = d(a, r) + 2$; if on the contrary a lies in G_q and b lies in G_r, then $d(u, r) = d(b, q)$ and $d(v, q) = d(a, r)$.)

Let V_{i-1} be the vertex set of H_{i-1} and let W_{i-1} be the set of leaves of vertices in V_{i-1}. Let H_i be the subgraph of $G \boxtimes G$ induced by the vertices in $V_{i-1} \cup W_{i-1}$. As observed above, H_i satisfies property (2). In order to show that H_i is isomorphic to T_i, we need only prove that (i) each vertex in W_{i-1} has a unique neighbour in V_{i-1} and (ii) W_{i-1} induces a stable set. Note that this also implies that H_i is an induced subgraph in $G \times G$ as $E(G \times G) \subseteq E(G \boxtimes G)$.

To show that Claim (i) is true, suppose on the contrary that the leaf $(v, u) \in W_{i-1}$ of some vertex $(a, b) \in V_{i-1}$ is adjacent to a vertex $(x, y) \in V_{i-1}$ different from (a, b). Up to symmetry we may assume that $a, u \in A$ and $b, v \in B$ and that a lies in G_r and b in G_q (the argument in the other cases is similar). We must have $x = a$, for otherwise x is either v or the child of v and $\ell_x = i$, which contradicts property (2) in H_{i-1}. Since $x \in A$, then $y \in B$ by property (2). Now, $y \neq b$, and y is a child of u. Now $d(y, q) = d(b, q) + 2$, whereas $d(x, r) = d(a, r)$, so $d(x, r) \neq d(y, q)$, a contradiction.

To show that Claim (ii) is true, suppose on the contrary that (a, b) and (b', a') are two adjacent vertices in W_{i-1}. We can consider $a, a' \in A$ and $b, b' \in B$ as they could not be adjacent otherwise. Let (s_a, s_b) and $(s_{b'}, s_{a'})$ be the supports of (a, b) and (b', a'), respectively. Note that $s_a, s_{a'} \in B$ and $s_b, s_{b'} \in A$, which implies that $\ell_{s_b} < \ell_{s_a}$ and $\ell_{s_{b'}} < \ell_{s_{a'}}$. By the definition of the labels, we have $\ell_{s_a} = \ell_a$ and $\ell_{s_{a'}} = \ell_{a'}$. Moreover, each of b and b' has label i and consequently has a child of label i and $\ell_a < \ell_b = i$. Thus, for (a, b) to be adjacent to (b', a'), a must be the neighbour of b' with label smaller than i, which is $s_{b'}$. In particular, $\ell_a = \ell_{s_{b'}}$, and, by a symmetric argument, $\ell_{a'} = \ell_{s_b}$. Putting this all together, we obtain that if (a, b) is adjacent to (b', a'), then $\ell_a = \ell_{s_{b'}} < \ell_{s_{a'}} = \ell_{a'} = \ell_{s_b} < \ell_{s_a} = \ell_a$ which is a contradiction. □

To prove Theorem 17 and Theorem 18, we study the graph H_k defined as follows. We start from the binomial tree T_k whose vertex set is partitioned into three sets X_1, X_2, X_3. The root of T_k is in X_1. For every $v \in X_1 \cup X_3$, the children of v are in X_2. For every $v \in X_2$ the children of v are placed according to the position of the parent w of v: if $w \in X_1$ then the children of v are in X_3; if $w \in X_3$ then the children of v are in X_1. Now H_k is obtained by adding to T_k all edges between X_1 and X_3.

Theorem 16. *For $k \geq 1$, $\Gamma(H_k) \leq 5$. Furthermore, for $k \geq 9$, $\Gamma(H_k) = 5$.*

Proof. We first observe that $\Gamma(H_k) \leq 6$ for every k. Indeed, in H_k every stable set is contained either in $A_1 = X_1 \cup X_2$ or in $A_2 = X_2 \cup X_3$. If H_k admits a greedy colouring with at least seven colours, then at least four colour classes are included in one of the two sets A_1 and A_2, say in A_j. This means that the subgraph H^* induced by A_j in H_k has Grundy number at least four. However, each component of H^* is a tree of radius at most 2, which implies that H^* has Grundy number at most 3.

In order to complete the first part of the theorem, let us give a more detailed analysis to show that $\Gamma(H_k) \leq 5$. The following two properties of T_k are useful.

(1) Any vertex $v \in X_2$ has either exactly one neighbour in X_1 or exactly one neighbour in X_3 (because if the parent of v is in one of X_1, X_3, then all its children are in the other of these two sets).

(2) For $i = 1, 3$, no path on five vertices in $X_i \cup X_2$ has its two endvertices in

X_i (because every component of $X_i \cup X_2$ consists of either the root of T_k and its children, or some vertex of X_2, its children and its grandchildren.)

Suppose that there exists a greedy 6-colouring φ on H_k.
Case 1: $\varphi(v) \in \{5,6\}$ for $v \in X_2$. Vertex v has neighbours of colours $1,2,3,4$. By property (1), v is adjacent to at most one vertex of X_1 or X_3. So there is $i \in \{1,3\}$ such that v has neighbours $w_1, w_2, w_3 \in X_i$ with $\varphi(w_1) < \varphi(w_2) < \varphi(w_3) \leq 4$. Then w_3 has a neighbour w_4 with $\varphi(w_4) = \varphi(w_2)$, and w_4 has a neighbour w_5 with $\varphi(w_5) = \varphi(w_1)$. Since $\{w_2, w_4\}$ and $\{w_1, w_5\}$ are stable sets, we have $w_4 \in X_2$ and $w_5 \in X_i$. But then the path w_1-v-w_3-w_4-w_5 contradicts property (2).
Case 2: $\varphi(v) = 6$ for some $v \in X_1 \cup X_3$. Let i be the index in $\{1,3\}$ such that $v \in X_i$. Vertex v has a neighbour w with $\varphi(w) = 5$. Then $w \in X_{4-i}$, otherwise Case 1 applies. Vertices v and w have neighbours u_v and u_w of colour 4, possibly $u_v = u_w$, but we cannot have one in X_1 and the other in X_3. Hence one vertex $u \in \{u_v, u_w\}$ is in X_2. Let t be its neighbour in $\{v, w\}$ and j the index such that $t \in X_j$. Vertex u has three neighbours a, b, c such that $\{\varphi(a), \varphi(b), \varphi(c)\} = \{1,2,3\}$. By property (1), either two elements of $\{a,b,c\}$, say a, b, are in X_j, or $\{a,b,c\} \subset X_{4-j}$. If $a, b \in X_j$, we may assume $\varphi(a) < \varphi(b)$, and we pick a neighbour d of t with $\varphi(d) = \varphi(b)$ and a neighbour e of d with $\varphi(e) = \varphi(a)$. Since $\{a, e\}$ and $\{b, d\}$ are stable sets in H_k, we have $d \in X_2, e \in X_j$. But then the path e-d-t-u-a contradicts property (2). If $\{a,b,c\} \subset X_{4-j}$, we may assume that $\varphi(a) = 1$, $\varphi(b) = 2$ and $\varphi(c) = 3$. There is a neighbour d of c with $\varphi(d) = 2$ and a neighbour e of d with $\varphi(e) = 1$. Since $\{a, e\}$ and $\{b, d\}$ are stable sets in H_k, we have $d \in X_2, e \in X_{4-j}$. But then the path e-d-c-u-a contradicts property (2). Thus we have shown that $\Gamma(H_k) \leq 5$, which completes the first part of the theorem.

Now, we show that $\Gamma(H_k) = 5$ when $k \geq 9$. We know that $\Gamma(T_k) = k$, so T_k contains a path a_1-a_2-$\cdots$-a_9 whose vertices are coloured $k, k-1, \dots, k-8$ respectively, where a_1 is the root of T_k, and a path a_2-b_3-b_4-b_5 whose vertices are coloured $k-1, k-3, k-4, k-5$, and a path a_6-b_7-b_8 whose vertices are coloured $k-5, k-7, k-8$. Note that vertices a_1, a_5, a_9, b_5 are in X_1, vertices $a_2, a_4, a_6, a_8, b_4, b_8$ are in X_2 and vertices a_3, a_7, b_3, b_7 are in X_3. Now we can make a greedy colouring of H_k with five colours, where vertices a_2, a_5, b_5, b_8, a_9 receive colour 1, vertices a_3, b_4, b_7, a_8 receive colour 2, vertices b_3, a_6 receive colour 3, and vertices a_1 and a_7 receive colours 4 and 5. $\square$

Theorem 17. *If G is a graph with at least one edge and $k \geq 1$, then $\Gamma(G \times H_k) \geq k$.*

Proof. It is enough to prove the theorem when $G = K_2$, $V(G) = \{v_1, v_2\}$. We claim that $\Gamma(G \times H_k) \geq k$. To see this, let $Y_i = \{v_1\} \times X_i$ for $i = 1, 3$ and $Y_2 = \{v_2\} \times X_2$. Then it is easy to check that $Y_1 \cup Y_2 \cup Y_3$ induces a copy of T_k in $K_2 \times H_k$, where Y_i plays the role of X_i in the partition of H_k. $\square$

Theorem 18. *If G is a connected non-complete graph and $k \geq 1$, then $\Gamma(G \boxtimes H_k) \geq k$.*

Proof. It is enough to prove the theorem when $G = P_3 = v_1\text{-}v_2\text{-}v_3$ as G contains an induced subgraph isomorphic to P_3. We claim that $\Gamma(G \boxtimes H_k) \geq k$. To see this, let $Y_i = \{v_i\} \times X_i$ for $i \in \{1, 2, 3\}$. It is easy to check that $Y_1 \cup Y_2 \cup Y_3$ induces a copy of T_k in $P_3 \boxtimes H_k$, where Y_i plays the role of X_i in the partition of H_k. $\square$

If G is a the disjoint union of complete graphs, then there is an upper bound on $\Gamma(G \boxtimes H)$ as a function of $\Gamma(G)$ and $\Gamma(H)$. It is enough to consider the case $G = K_{m+1}$. Observe that $K_{m+1} \boxtimes H = H[K_{m+1}]$. Hence by Theorem 1 we get the following.

Proposition 19. *If $\Gamma(H) = k \geq 2$ and $m \geq 1$ then $\Gamma(K_{m+1} \boxtimes H) \leq m2^{k-1} + k$.*

4 Comments and open questions

Section 3 shows that any upper bound on the Grundy number of $G \times H$ as a function of $\Delta(G), \Gamma(H)$ is possible only if $\Gamma(H) \leq 4$. Perhaps a good test case is to decide whether $\Gamma(K_2 \times H)$ is bounded for $\Gamma(H) \leq 4$. (On the other hand, if the maximum degree of both graphs may intervene, then we know the easy inequality $\Gamma(G \times H) \leq \Delta(G \times H) + 1 \leq \Delta(G)\Delta(H) + 1$, but this is probably not a very interesting bound.)

A referee asked whether, in replacement for the now failed Conjecture 4, the following inequality could be conjectured to hold for any two graphs G and H.

$$\Gamma(G \Box H) \leq \max\{(\Delta(G) + 1)\,\Gamma(H), (\Delta(H) + 1)\,\Gamma(G)\} \quad (**)$$

We can prove this inequality, as follows. First suppose that $\Gamma(G) = 1$. Then G has no edge, so the left-hand side of $(**)$ is $\Gamma(H)$ and the right-hand side is $\max\{\Gamma(H), \Delta(H) + 1\} = \Delta(H) + 1$, so $(**)$ holds for every H. Now suppose that $\Gamma(G) \geq 2$ and similarly $\Gamma(H) \geq 2$. On the right-hand side of $(**)$, we have $\max\{(\Delta(G) + 1)\,\Gamma(H), (\Delta(H) + 1)\,\Gamma(G)\} \geq \frac{1}{2}\{(\Delta(G) + 1)\,\Gamma(H) + (\Delta(H) + 1)\,\Gamma(G)\} \geq \Delta(G) + 1 + \Delta(H) + 1$. On the left-hand side we have $\Gamma(G \Box H) \leq \Delta(G \Box H) + 1 = \Delta(G) + \Delta(H) + 1$, so it is strictly smaller than the right-hand side. Actually this proof shows that $(**)$ tends to give a weak upper bound on $\Gamma(G \Box H)$ in general; indeed in all cases it is weaker than $\Delta(G \Box H) + 1$.

Concerning the lexicographic product, it was proved in [1] that if $\Gamma(H) = k$, then for any graph G, we have $\Gamma(G[H]) = \Gamma(G[K_k])$. Moreover, as mentioned in Remark 11, we have $\Gamma(G[K_k]) = \Gamma(G[S_k] \Box K_k)$. So $\Gamma(G[H]) = \Gamma(G[S_k] \Box K_k)$. Thus the Grundy number of the lexicographic product of any two graphs G and H can be seen as a particular case of the Grundy number of the Cartesian product of two graphs. Therefore we feel that the most interesting questions in this domain are about the Cartesian product. In particular, although Conjecture 4 is now known to be false because of Corollary 12, one may still wonder whether there exists a constant λ such that any two graphs G and H satisfy $\Gamma(G \Box H) \leq \lambda\,(\Delta(G) + 1)\,\Gamma(H)$. Note that the graph H given in the proof of Corollary 12 satisfies $\Gamma(K_2 \Box H) = \frac{7}{6}(\Delta(K_2) + 1)\,\Gamma(H)$, and so does the second graph H'. We could not find a graph with a ratio larger than $\frac{7}{6}$. Is it true that $\Gamma(K_2 \Box H) \leq 2c\,\Gamma(H)$ for some constant $c \geq 7/6$?

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APÊNDICE B

ON THE b-COLORING OF GRAPHS WITH FEW P_4s

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Abstract. A b-coloring of a graph is a coloring such that every color class contains a vertex adjacent to at least one vertex receiving each of the colors not assigned to it. The b-chromatic number, denoted by $\chi_b(G)$, is the maximum number t such that G admits a b-coloring with t colors. It is known that determining the b-chromatic number is an NP-Hard problem [11]. In 2009, Bonomo et al. [5] proved that determining the b-chromatic number for P_4-sparse graphs is polynomial-time solvable. In this work, we generalize this result for $(q, q-4)$-graphs, for every fixed q. These are graphs for which no set of at most q vertices induces more than $q-4$ distinct P_4's.

1. Introduction

Let G be a finite undirected graph, without loops or multiple edges. Given a coloring of G, a vertex v is said to be dominant if v is adjacent to at least one vertex colored within each of the colors not assigned to v. A b-coloring of G is a coloring such that every color class contains a dominant vertex. If a proper coloring c of G is not a b-coloring, there exists one color (say *blue*) without a dominant vertex. In this case, we can obtain another coloring of G based on c by coloring every blue vertex with another color of c. So we can decrease the number of colors until we obtain a b-coloring. Using this procedure, what is the worst case? What is the largest number of colors such that we have a b-coloring?

The b-chromatic number $\chi_b(G)$ of a graph G is the maximum number t such that there exists a b-coloring of G with t colors. This parameter has been introduced by R. W. Irving and D. F. Manlove [11]. They proved that determining the b-chromatic number is polynomial-time solvable for trees, but is NP-hard for general graphs. In [18], Kratochv, Tuza and Voigt proved that computing the b-chromatic number is NP-hard even if G is a connected bipartite graph. Recently, some related concepts concerning the b-coloring problem has been studied, like b-continuity and b-monotonicity (see [10, 16, 17, 19] and references therein).

In 2004, the b-coloring problem was investigated for cographs [17], but they did not prove the NP-hardness for this graph class. Many NP-hard problems were proved to be polynomial time solvable on cographs, but it is known that computing the a-chromatic number of a cograph is NP-hard [4]. Cographs are graphs such that no 4-vertex subset induces a P_4 (a path with four vertices) and were first defined in [6]. These graphs have a really nice structure.

Proposition 1.1. [6] *Every induced subgraph of a cograph is also a cograph. If G is a non-trivial cograph, then either G or $\bar{G}$ is not connected, where $\bar{G}$ is the complement of G.*

In 2005, the b-coloring problem was investigated for P_4-sparse graphs [9], but they also did not prove the NP-hardness for this graph class. A graph is P_4-sparse if every 5-vertex subset induces at most one P_4. This graph class was introduced in [8]. They generalize cographs and can be recognized in linear time [13]. P_4-sparse graphs have also a nice structural description, similar to cographs.

Proposition 1.2. [8, 12] *If G is a non-trivial P_4-sparse graph, then either G or $\bar{G}$ is not connected, or G is a spider. A* spider *is a graph whose vertex set can be partitioned into S, C and R, where $S = \{s_1, \ldots, s_k\}$ and $C = \{c_1, \ldots, c_k\}$ for $k \geq 1$ are respectively a stable set and a complete set; s_i is adjacent to c_i if and only if $i = j$ (a thin spider), or s_i is adjacent* *to c_i if and only if $i \neq j$ (a thick spider); and every vertex in R is adjacent to all vertices in C and non-adjacent to all vertices in S.*

Finally, in 2009, using these nice decompositions, Bonomo, Dur Maffray, Marenco and Valencia-Pabon [5] proved that determining the b-chromatic number for P_4-sparse graphs (and, in particular, for cographs) is polynomial-time solvable. They also have presented a dynamic programming polynomial-time algorithm to compute the b-chromatic number of a P_4-sparse graph.

In this work, we generalize this result for several graph classes. Babel and Olariu [2] defined a graph as $(q, q-4)$-*graph* if no set of at most q vertices induces more than $q-4$ distinct P_4's. For example, cographs and P_4-sparse graphs are precisely $(4, 0)$-graphs and $(5, 1)$-graphs respectively. The C_5-free P_4-extendible graphs coincides with the $(6, 2)$-graphs and the P_4-lite graphs are special $(7, 3)$-graphs.

Theorem 1.3 (Main result)**.** *Let $q > 0$ be an integer fixed. The b-chromatic number of a $(q, q-4)$-graph G can be computed in polynomial time.*

This paper is structured as follows. Section 2 contains structural results for $(q, q-4)$-graphs and the primeval decomposition, based on p-connected graphs. We also provide a decomposition lemma for $(q, q-4)$-graphs, based on Babel et al. [3]. Section 3 presents the results of Bonomo et al. [5], including the definition of *dominance vector* and its relations with union, join and spider operations. It is a fact that, once calculated the dominance vector, we have the b-chromatic number. We also present a technical lemma that helps us to calculate the dominance vector with another operation, that we call *small*. With this, we have the proof of our main theorem above. Finally, Section 4 is devoted to show a proof sketch of this technical lemma.

2. Decomposing $(q, q-4)$-graphs

The structural descriptions of Propositions 1.1 and 1.2 gives us known decomposition trees for cographs and P_4-sparse graphs. A decomposition rooted tree of a graph G is a tree T_G, where each non-leaf node in T_G is labeled with an operation over graphs and represents the subgraph of G obtained by applying its operation to its children. The root node of T_G represents the original graph G.

The *union operation* of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$, where $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *join operation* of two graphs G_1 and G_2 is the graph $G_1 \vee G_2$, where $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup V(G_1) \times V(G_2)$. A join operation is a union operation plus the inclusion of all possible edges between G_1 and G_2.

The *spider operation* over three vertex sets (S, C, R) is defined in Proposition 1.2. We may visualize S, C and R respectively as the legs, the body and the head of the spider. Note that R is allowed to be empty (in this case, we say that the resulting graph G is a *headless spider*). Note also that a complement of a thin spider is a thick spider. Given a spider, its spider partition (S, C, R) can be found in linear time [12]. A spider node of a decomposition tree can be viewed as the vertices in S and C being leaves and R being a tree node that represents its induced subgraph.

Proposition 1.1 implies that, to each cograph G, there exists a decomposition tree T_G with only two operations (union and join) and each leaf is a vertex of G. This tree can be computed in polynomial time [7].

Proposition 1.2 implies that, to each P_4-sparse graph G, there exists a decomposition tree T_G with only three operations (union, join and spider) and each leaf is a vertex of G [12].

Using these nice decompositions, Bonomo et al. [5] have presented a polynomial time algorithm to compute the b-chromatic number of a cograph or P_4-sparse graph.

Babel and Olariu [2] called a graph as $(q, q-4)$-*graph* if no set of at most q vertices induces more than $q-4$ distinct P_4's. They also proved the existence of a decomposition tree for $(q, q-4)$-graphs, whose leaves are special graphs, called p-connected graphs.

Definition 2.1. *A graph G is* p-connected *if, for every partition of $V(G)$ into nonempty disjoint sets A and B, there exists a P_4 which is (A, B)-crossing, that is, a P_4 containing vertices from both A and B. A p-connected graph G is* separable *if there exists a partition of $V(G)$ into nonempty disjoint subsets A and B such that each (A, B)-crossing P_4 has its* midpoints *in A and its* endpoints *in B. A maximal p-connected induced subgraph is called a* p-component*. Vertices which are not contained in a nontrivial p-component are called* weak*.*

Theorem below from Jamison and Olariu [14] is an important and general structure result for arbitrary graphs, using p-connected graphs.

Theorem 2.2 (p-connected structure theorem [14])**.** *Given a graph $G = (V, E)$, exactly one of the following statements holds:*

(i) *G is disconnected;*
(ii) *$\bar{G}$ is disconnected;*
(iii) *G contains a unique proper separable p-component H with separation (H_1, H_2) such that every vertex outside H is adjacent to every vertex in H_1 and to no vertex in H_2;*
(iv) *G is p-connected.*

This theorem gives us a decomposition tree for general graphs, called *primeval decomposition tree*, which can be computed in polynomial time [14]. The leaves of its decomposition tree are p-connected graphs (see (iv)) and its operations are *union* (see (i)), *join* (see (ii)) or the operation defined by (iii).

It was proved in [2] that every p-connected $(q, q-4)$-graph is a headless spider or has less than q vertices. So, if the p-component H of (iii) has less than q vertices, we have another operation that we call *small*(q) *operation*.

Definition 2.3. *Given $q > 0$, we define the* small(q) operation *of two graphs $(G_M, H = (H_1, H_2))$ as the graph G obtained by joining all edges between G_M and H_1, where H is a graph with less than q vertices, H_1 is an induced subgraph of H and H_2 is $G[V(G) \setminus V(H_1)]$.*

Observe that the vertex set M of G_M is a module of G. The next auxiliary lemma gives us a nice decomposition tree for $(q, q-4)$-graphs. It is important to note that, in [3], they mentioned a very similar decomposition tree for $(q, q-4)$-graphs, using the operation defined by (iii). With operations spider and small, instead of (iii), we can obtain smaller leaves and use Bonomo et al. [5] results for spider operation.

Lemma 2.4. *Let $q > 0$ be an integer fixed and let G be a $(q, q-4)$-graph. Then, there exists a decomposition tree of G with operations union, join, spider and small(q), whose leaves are graphs with less than q vertices. Such decomposition tree can be computed in linear time.*

Proof. Remember that every induced subgraph of a $(q, q-4)$-graph is also a $(q, q-4)$-graph. So, apply the primeval decomposition of Theorem 2.2 on G. If (i), then we have the union operation of induced subgraphs of G. If (ii), then we have the join operation of induced subgraphs of G.

If (iii), then the p-component H is a headless spider or has less than q vertices. Remember that every p-connected $(q, q-4)$-graph satisfies this [2]. If the p-component H in (iii) is a headless spider, then it is easy to see that G is a spider. So, we have that the operation defined by (iii) is a spider operation.

If the p-component H in (iii) is not a headless spider, then it is a small graph with less than q vertices. So, we have that the operation defined by (iii) is the small(q) operation.

If (iv), then G is a $(q, q-4)$ p-connected graph, which again can be a headless spider or a small graph with less than q vertices. In the first case, we have again the spider operation. In the other case, we have a leaf, that is a small graph. □

3. b-coloring $(q, q-4)$-graphs

In [5], Bonomo et al. show how to obtain the b-chromatic number of a cograph or P_4-sparse graph in poynomial time. For this, they introduced the *dominance vector* of a graph.

Definition 3.1. *Let G be a graph. Given a coloring of G, a vertex v is said to be dominant if v is adjacent to at least one vertex colored within each of the colors not assigned to v. The* dominance vector *dom_G of G is such that $dom_G[t]$ is the maximum number of distinct color classes admitting dominant vertices in any coloring of G with t colors. where $\chi(G) \leq t \leq |V(G)|$.*

Note that a graph G admits a b-coloring with t colors if and only if $dom_G[t] = t$. So, the b-chromatic number $\chi_b(G)$ is the maximum number t such that $dom_G[t] = t$. Thus, once calculated the dominance vector of a graph, we have its b-chromatic number. Bonomo et al. [5] proved that calculating the dominance vector is polynomial-time solvable for cographs and P_4-sparse graphs.

Lemmas 3.2 and 3.3 below from [5] show how to obtain the dominance vector for union, join and spider operations. The calculation of $\chi(G)$ is from [7] and [15].

Lemma 3.2 (Dominance vector for union and join operations [5])**.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $V_1 \cap V_2 = \emptyset$ and let $t \geq \chi(G)$. If $G = G_1 \cup G_2$, then $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$ and*

$$dom_G[t] \;=\; min\{t, dom_{G_1}[t] + dom_{G_2}[t]\}.$$

If $G = G_1 \vee G_2$, let $a = max\{\chi(G_1), t - |V(G_2)|\}$ and $b = \min\{|V(G_1)|, t - \chi(G_2)\}$. Then, $\chi(G) = \chi(G_1) + \chi(G_2)$ and

$$dom_G[t] \;=\; \max_{a \leq j \leq b}\{dom_{G_1}[t] + dom_{G_2}[t-j]\}.$$

Lemma 3.3 (Dominance vector for spider operations [5])**.** *Let G be a spider with partition (S, C, R), where $k = |S| = |C| \geq 2$. If R is empty, consider $\chi(G[R]) = 0$ and $dom_{G[R]}[0] = 0$. Thus, $\chi(G) = k + \chi(G[R])$ and*

(a) *If G is a thin spider, then*

$$dom_G[i] \;=\; \begin{cases} k + dom_{G[R]}[i-k], & \text{if } k + \chi(G[R]) \leq i \leq k + |R|, \\ k, & \text{if } i = k + |R| + 1, \\ 0, & \text{if } i > k + |R| + 1 \end{cases}$$

(b) *If G is a thick spider, then*

$$dom_G[i] \;=\; \begin{cases} k + dom_{G[R]}[i-k], & \text{if } k + \chi(G[R]) \leq i \leq k + |R|, \\ \min\{k, 4k - 2i + 2|R|\}, & \text{if } k + |R| + 1 \leq i \leq 2k + |R|, \\ 0, & \text{if } i > 2k + |R| \end{cases}$$

Using these lemmas, Bonomo et al. proved theorem below.

Theorem 3.4 (Bonomo et al. [5])**.** *The dominance vector and the b-chromatic number of a cograph or P_4-sparse graph can be computed in $O(n^3)$ time.*

Lemma 2.4 presents a decomposition tree for $(q, q-4)$-graphs using four operations: *union*, *join*, *spider* and *small*(q). Lemmas 3.2 and 3.3 show how to obtain the chromatic number and the dominance vector for three of them. If we can calculate the chromatic number and the dominance vector for the small(q) operation, then we are finished. This is our main technical lemma, which is proved in Section 4.

Lemma 3.5. *Let $q > 0$ be an integer fixed, let H be a small graph with less then q vertices and let H_1 and H_2 be induced subgraphs of H such that a vertex partition of H. Given a graph G_M with n vertices, let G be the graph obtained by applying small(q) operation over $(G_M, H = (H_1, H_2))$ (just join all edges between G_M and H_1). Then, given the chromatic number $\chi(G_M)$ and the dominance vector dom_M of G_M, we can calculate the chromatic number $\chi(G)$ in time $\Theta(n)$ and the dominance vector dom_G of G in time $\Theta(n^2)$.*

This result is intuitive because H is a small graph with constant size q. So it is possible to apply several colorings on H and make several calculations in constant time, depending only on q, and not on n. In [1], it was proved that the chromatic number of a $(q, q-4)$-graph can be calculated in linear time using the primeval decomposition. So, our hard work is to calculate the dominance vector. With this lemma, we have the proof of our main result.

Proof of Theorem 1.3. Follows directly from Theorem 2.2 and Lemmas 2.4, 3.2, 3.3 and 3.5 as explained. From the dominance vector of G, the b-chromatic number is the maximum t such that $dom_G[t] = t$. □

4. b-coloring small(q) operation

This section is devoted to prove Lemma 3.5. Let $G = (V, E)$ be a graph and M be a module of G (the neighborhoods outside the module of the vertices within the module are all equal). Let $G_M = G[M]$ and let $N(M)$ be the neighborhood of a vertex in M. Let H, H_1 and H_2 be the subgraphs of G induced by $V \setminus M$, $N(M)$ and $V(H) \setminus N(M)$, respectively. If H has less than q vertices, G is obtained by applying small(q) operation over $(G_M, H = (H_1, H_2))$.

To calculate $dom_G[t]$, auxiliary lemma below shows us that there exists a good coloring such that: (a) all colors appears in M or H_1 or (b) vertices of M have distinct colors. Given a coloring C of G and a subgraph G' of G, let $n(C)$ be the number of colors used in C and let (C, G') be the restriction of the coloring C to G'.

Lemma 4.1. *If $\chi(G) \leq t \leq |V(G)|$, then there is a proper coloring C of G with t colors that maximizes the number of color classes with dominant vertices such that $n(C) = n(C, H_1) + n(C, G_M)$ or $n(C, G_M) = |V(G_M)|$.*

Proof. Let C be a coloring of G with t colors that maximizes the number of color classes with dominant vertices and then maximizes $n(C, G_M)$. Since M is a module, each vertex in G_M is adjacent to all vertices in H_1. Thus, $n(C) \geq n(C, H_1) + n(C, G_M)$. Suppose that C does not satisfy the lemma. Since $n(C) > n(C, H_1) + n(C, G_M)$, then there is a color c that appears only in vertices of H_2 and thus no vertex of G_M is dominant in C. Since $n(C, G_M) < |V(G_M)|$, then there are two vertices v and v' of G_M that have the same color in C. Consider the coloring C' obtained from C by coloring v with color c. Note that any dominant vertex in C is also a dominant vertex in C' and thus C' also has a maximum number of color classes with dominant vertices among colorings with t colors. Note that $n(C', G_M) > n(C, G_M)$. Suppose again that C' does not satisfy the lemma. So, we can repeat this argument until we obtain a coloring C^* such that all vertices of G_M are colored with distinct colors. Thus, $n(C^*, G_M) = |V(G_M)|$ as desired. □

Applying this lemma, we have four possible cases:

- (a) all colors appears in M or H_1

  - (a.1) There is no dominant vertex in H_2
  - (a.2) There is a dominant vertex in H_2
- (b) Vertices of M have distinct colors
  - (b.1) There are colors in M that are not in H
  - (b.2) Every color in M appears in H

Case (b.2) is easy to handle because it implies that $|M| \leq |V(H)|$. Since we will force that $|V(H)| \leq q$, we can obtain all colorings of G with t colors in constant time. To deal with cases (a.1), (a.2) and (b.1), we have to define some parameters.

Let $\mathcal{C}(t)$ be the set of all colorings of H with t colors and let $\mathcal{C}(t,t')$ be subset of $\mathcal{C}(t)$ with colorings of H such that H_1 uses t' colors. Let $C \in \mathcal{C}(t,t')$. For $H' \subseteq H$, let $c(C,H')$ denote the set of colors used in H'. We say that a vertex v in H_1 is partially dominant if v is adjacent to at least one vertex receiving each color in $c(C,H_1)$. Let $d_1(C)$ be the number of colors classes of C with partially dominant vertices in H_1. Let $d_2(C)$ be the number of color classes of $c(C,H_2) \setminus c(C,H_1)$ with a dominant vertex. Let $d_3(C)$ be the number of color classes in $c(C,H_1)$ with either a dominant vertex in H_2 or a partially dominant vertex in H_1. Let $J \subseteq c(C,H_2) \setminus c(C,H_1)$. We say that a vertex v in H_1 is $\bar{J}$-dominant if v is adjacent to at least one vertex receiving each color in $c(C,H) \setminus J$. Let $d_4(C,J)$ be the number of color classes of C with either a dominant vertex in H_2 or a $\bar{J}$-dominant vertex in H_1 and $d_5(C,j) = \sup\{d_4(C,J) \mid J \subseteq c(C,H_2) \setminus c(C,H_1), |J| = j\}$.

Let $\chi(G) \leq t \leq |V|$, let $t_1 = \max\{t - |V(G_M)|, 0\}$, let $t_2 = \min\{|V(H_1)|, t - \chi(G_M)\}$, let $t_3 = \min\{|V(H)|, t\}$, let $t_4 = \min\{t - |V(G_M)|, |V(H_1)|\}$ and let

$$
\begin{aligned}
\tau_1(t) &= \sup_{\substack{t_1 \leq t' \leq t_2 \\ t' \leq \hat{t} \leq t_3}} \{dom_{G_M}[t-t'] + d_1(C) \mid C \in \mathcal{C}(\hat{t},t')\} \\
\tau_2(t) &= \sup_{t_1 \leq t' \leq t_2} \{\min\{t-t', d_2(C) + dom_{G_M}[t-t']\} + d_3(C) \mid C \in \mathcal{C}(t,t')\} \\
\tau_3(t) &= \sup_{\substack{t_1 \leq \hat{t} \leq t_3 \\ 0 \leq t' \leq t_4}} \{d_5(C, \hat{t} + |V(G_M)| - t) \mid C \in \mathcal{C}(\hat{t},t')\}
\end{aligned}
$$

Excluding case (b.2) by forcing that $|V(G)| > 2|V(H)|$, we have the important lemma below.

Lemma 4.2. *If $\chi(G) \leq t \leq |V(G)|$ and $|V(G)| > 2|V(H)|$, then*

$$dom_G[t] = \max\{\tau_1(t), \tau_2(t), \tau_3(t)\}.$$

Proof. Let C be a coloring of G with t colors that maximizes the number of color classes with dominant vertices. According to Lemma 4.1, suppose that either $n(C,H_1) + n(C,G_M) = t$ or $n(C,H_1) + n(C,G_M) < t$ and $n(C,G_M) = |V(G_M)|$. Let $\hat{t} = n(C,H)$.

The first case considered is (a) when $n(C,H_1) + n(C,G_M) = t$. Note that if v is a dominant vertex in C, then v is dominant in (C,G_M) if $v \in V(G_M)$ and v is partially dominant in (C,H) if $v \in V(H_1)$. Let $t' = n(C,H_1)$. Since $\chi(G_M) \leq n(C,G_M) = t - n(C,H_1) \leq |V(G_M)|$, then $t - |V(G_M)| \leq t' \leq t - \chi(G_M)$. We also get that $t' \leq |V(H_1)|$ and, thus, $t_1 \leq t' \leq t_2$.

Now, consider (a.1) that there is no dominant vertex in H_2. In this case, $t' \leq \hat{t} \leq \min\{|V(H)|, t\}$. We also have that the number of color classes of C with dominant vertices of colors that appear in H_1 is precisely $d_1(C,H)$ and the with dominant vertices of colors that appear in G_M is at most $dom_{G_M}[t-t']$. Thus, if $n(C,H_1) + n(C,G_M) = t$ and there is no dominant vertex of C in H_2, then $dom_G[t] \leq \tau_1(t)$.

Now, consider (a.2) that there is at least one dominant vertex u in H_2. Since u is adjacent to every other color of C and every neighbour of u is in H, then $\hat{t} = t$. Note that the number of color classes of C with dominant vertices of colors that appear in H_1 is precisely $d_3(C,H)$.

The number of color classes of C with dominant vertices of colors that appear in G_M is at most $\min\{t-t', d_2(C)+dom_G[t-t']\}$. Thus, if $n(C,H_1)+n(C,G_M)=t$ and there is at least one dominant vertex of C in H_2, then $dom_G[t] \leq \tau_2(t)$.

The second case considered is (b) when $n(C,H_1)+n(C,G_M) < t$ and $n(C,G_M) = |V(G_M)|$. If (b.2) $c(C,G_M) \subseteq c(C,H)$, then $n(C,G_M) = |V(G_M)|$ implies that $|M| \leq |V(H)|$ and $|V(G)| \leq |V(H)|$, a contradiction. Thus, (b.1) there is a color unique to vertices in G_M. Note also that $n(C,H_1)+n(C,G_M) < t$ implies that there is a color unique to vertices in H_2. Thus, all dominant vertices of C are in H_1. Note that $\hat{t} \leq |V(H)|$ and $\hat{t} \leq t$ and, thus, $\hat{t} \leq t_3$. We also have that $t \leq \hat{t} + |V(G_M)|$ which implies that $\hat{t} \geq t_1$. Let $J = c(C,G_M) \cap c(C,H)$. Note that $t = \hat{t} + |V(G_M)| - |J|$ which implies that $|J| = \hat{t} + |V(G_M)| - t$. Since J is a subset of $c(C,H) \setminus c(C,H_1)$, then $|J| = \hat{t} + |V(G_M)| - t \leq \hat{t} - t'$ which implies that $t' \leq t - |V(G_M)|$. Since H_1 has at most $V(H_1)$ colors, then $t' \leq t_4$. Now, note that every dominant vertex of C is a $\bar{J}$-dominant vertex of H_1 in (C,H). Thus, the number of color classes with dominant vertices in C is $d_4((C,H),J)$, which is at most $d_5((C,H),\hat{t}+|V(G_M)|-t)$. Thus, if $n(C,H_1)+n(C,G_M) < t$ and $|V| > 2|V(H)|$, then $dom_G[t] \leq \tau_3(t)$.

We can conclude from the previous paragraphs that if $|V| > 2|V(H)|$, then $dom_G[t] \leq \max\{\tau_1(t), \tau_2(t), \tau_3(t)\}$. To conclude this proof, it remains to prove that $dom_G[t] \geq \tau_i(t)$, for $i \in \{1,2,3\}$. Let C_H be a coloring of H with $\hat{t}$ colors and $t' = n(C_H,H_1)$. We break into cases depending on C_H being related to each of the parameters $\tau_i(t)$. To do so, let C_M be a coloring of G_M with $t-t'$ colors and $dom_{G_M}[t-t']$ color classes with dominant vertices and C'_M be a coloring of G_M with $|V(G_M)|$ colors.

Suppose that $t_1 \leq t' \leq t_2$. If $\hat{t} \leq t$, then rename the colors in $c(C_H,H_2) \setminus c(C_H,H_1)$ to colors in the set $c(C_M)$ and let C be the coloring of G obtained by piecing together this coloring with C_M. Note that C has precisely t colors and there are $dom_{G_M}[t-t']$ color classes with dominant vertices in colors of $c(C,G_M)$ and $d_1(C_H)$ color classes with dominant vertices in colors of $c(C,H_1)$. Since $c(C,G_M) \cap c(C,H_1) = \emptyset$, then C has at least $dom_{G_M}[t-t'] + d_1(C)$ color classes with dominant vertices. This implies that $dom_G[t] \geq \tau_1(t)$.

Now, suppose that $\hat{t} = t$. Let $c(C_M) = \{c_1, \ldots, c_{t-t'}\}$ and suppose that the color classes with indices in $\{1, \ldots, dom_{G_M}[t-t']\}$ have dominant vertices in C_M. Now let C'_H be obtained from C_H by renaming the colors in $c(C_H,H_2) \setminus c(C_H,H_1)$ to colors in $c(C_M)$ in such a way that the color classes with the highest indices have dominant vertices. Note that this is possible as $c(C_H,H_2) \setminus c(C_H,H_1)$ has size precisely $t-t'$. Let C be obtained by piecing together the colorings C_M and C'_H. Note that C has precisely t colors, $d_3(C_H)$ color classes in $c(C,H_1)$ with dominant vertices and $\min\{t-t', d_2(C)+dom_G[t-t']\}$ color classes in $c(C,G_M)$ with dominant vertices. This implies that $dom_G[t] \geq \tau_2(t)$.

Now, suppose that $0 \leq t' \leq t_4$ and $t_1 \leq \hat{t} \leq t_3$. Note that $0 \leq \hat{t} + |V(G_M)| - t \leq \hat{t} - t'$, as $\hat{t} \geq t_1 = t - |V(G_M)|$ and $t' \leq t_4 \leq t - |V(G_M)|$. Thus, let J be a subset of $c(C_H,H_2) \setminus c(C_H,H_1)$ such that $d_4(C_H,J) = d_5(C_H, \hat{t}+|V(G_M)|-t)$. Let C'_H be obtained by renaming the colors of C_H in the set J to colors in C'_M so that $|c(C'_H) \cap c(C'_M)| = |J| = \hat{t} + |V(G_M)| - t$. Let C be obtained by piecing together the colorings C'_H and C'_M. Note that $n(C) = n(C,H) + n(C,G_M) - |J| = \hat{t} + |V(G_M)| - |J| = t$. This implies that $dom_G[t] \geq \tau_3(t)$. □

Proof of Lemma 3.5. Since $|V(H)| \leq q$, where q is an integer fixed, we have that parameters $\tau_1(t)$, $\tau_2(t)$ and $\tau_3(t)$ can be obtained in linear time (once fixed t' and $\hat{t}$, the value in sup can be obtained in constant time that depends only on q). If $|V(G)| \leq 2|V(H)| \leq 2q$, then we can calculate $dom_G[t]$ in constant time. If $|V(G)| > 2|V(H)|$, then, applying Lemma 4.2, we have $dom_G[t]$ in linear time. So we can obtain the dominance vector dom_G of G in time $\Theta(n^2)$ for all possible values of t. □

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APÊNDICE C

b-colouring cacti☆

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Abstract

A b-colouring of a graph is a colouring of its vertices such that every colour class contains a vertex that has a neighbour in all other classes. The b-chromatic number of a graph is the largest integer k such that the graph has a b-colouring with k colours. We show here how to compute in polynomial time the b-chromatic number of a cactus. This generalizes the seminal result of Irving and Manlove [1] on trees.

Keywords: b-chromatic number, b-colouring, cactus, m-degree, exact algorithm.

1. Introduction

Let G be a simple graph. A *proper colouring of* G is an assignment of colours to the vertices of G such that no two adjacent vertices have the same colour. The *chromatic number of* G is the minimum integer $\chi(G)$ such that G has a proper colouring with $\chi(G)$ colours. Suppose that we have a proper colouring of G and there exists a colour c such that every vertex v with colour c is not adjacent to at least one other colour (which may depend on v); then we can change the colour of these vertices and thus obtain a proper colouring with fewer colours. This heuristic can be applied iteratively, but we cannot

☆Supported by CAPES,CNPq/Brazil and CNRS/France.

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expect to reach the chromatic number of G, since the colouring problem is $\mathcal{NP}$-hard. On the basis of this idea, Irving and Manlove introduced the notion of b-colouring in [1]. Intuitively, a b-colouring is a proper colouring that cannot be improved by the above heuristic, and the b-chromatic number measures the worst possible such colouring. More formally, let G be any graph, given with a colouring of its vertices. A vertex u is said to be a *b-vertex* (for this colouring) if u has a neighbour coloured with each colour different from the colour of u. A *b-colouring* of G is a proper colouring of G such that each colour class contains a b-vertex. The *b-chromatic number* of G is the largest integer k such that G has a b-colouring with k colours. We denote this value by $\chi_b(G)$. In a b-colouring with k colours, let v_i be any b-vertex of colour i ($i = 1, \ldots, k$); then we say that the set $\{v_1, \ldots, v_k\}$ is a *basis* of the b-colouring. A b-colouring may have many bases.

Naturally, a proper colouring of G with $\chi(G)$ colours is a b-colouring of G, since it cannot be improved. So, $\chi(G) \leq \chi_b(G)$. For an upper bound, observe that if G has a b-colouring with k colours, then G has at least k vertices with degree at least $k-1$ (the vertices of any basis of the b-colouring). So, if $m(G)$ is the largest integer such that G has at least $m(G)$ vertices with degree at least $m(G) - 1$, we know that G cannot have a b-colouring with more than $m(G)$ colours, i.e.,

$$\chi_b(G) \leq m(G).$$

This upper bound was introduced by Irving and Manlove in [1]. They showed that the difference between $\chi_b(G)$ and $m(G)$ can be arbitrarily large for general graphs. They proved that $\chi_b(G)$ is equal to $m(G)$ or $m(G) - 1$ when G is a tree, and provided a polynomial time algorithm that computes $\chi_b(G)$ for every tree. In addition, the problem was proved to be NP-hard [1], even when restricted to bipartite graphs [2].

The fact that the difference between $\chi_b(G)$ and $m(G)$ can be arbitrarily large for a general graph, but is at most one for trees, made us wonder what kind of graphs have the same property. A natural way is to investigate "tree-like" graphs, i.e., graphs that have a tree structure, such as cacti, block graphs, k-trees, etc. Here we consider the cacti. A graph G is a *cactus* if every two cycles of G intersect in at most one vertex. Our main result is the following:

Theorem 1. *Let G be a cactus with $m(G) \geq 7$. Then $\chi_b(G)$ is equal to either $m(G)$ or $m(G) - 1$. Moreover, we can determine the value of $\chi_b(G)$*

(and a b-colouring with $\chi_b(G)$ colours) in polynomial time.

Here is an outline of the proof of Theorem 1. We first present a family $\mathcal{F}$ of cacti that cannot be b-coloured with $m(G)$ colours and show how to b-colour a cacti $G \in \mathcal{F}$ with $m(G) - 1$ colours, thus proving that $\chi_b(G) = m(G) - 1$. Then, we prove that a cactus G not in $\mathcal{F}$ has a special set of vertices, that we call a quasi-good set; also, we give a polynomial-time algorithm that finds such a set. Finally, we prove that if a cactus G has a quasi-good set W and $m(G) \geq 7$ (hence, $G \notin \mathcal{F}$), then W is the basis of a b-colouring of G with $m(G)$ colours.

In Section 2, we give the necessary definitions and present the notation that will be used; in Section 3, we present a family $\mathcal{F}$ of cacti that cannot be b-coloured with $m(G)$ colours; in Section 4, we show how to obtain a quasi-good set of G, if one exists (this algorithm works at the same time as a recognition algorithm for graphs in $\mathcal{F}$); in Section 5, we show how to obtain a b-colouring of G with $m(G) - 1$ colours, in the case where $G \in \mathcal{F}$; and, finally, in Section 6, we show how to obtain a b-colouring of G with $m(G)$ colours, in the case where G has a quasi-good set.

2. Definitions and notation

Let G be a graph; we denote by $V(G)$ and $E(G)$ the sets of vertices and edges of G, respectively (or simply V and E, if there is no ambiguity). Let $u \in V(G)$; the *neighbourhood* of u in G is the set $N(u) = \{v \in V(G) \mid (u, v) \in E(G)\}$, the *closed neighbourhood of* u is the set $N[u] = N(u) \cup \{u\}$, and the *degree of* u is $d(u) = |N(u)|$. If $v \in N(u)$, we say that u is *adjacent* to v. If $X \subseteq V(G)$ and there is no ambiguity, then $N^X(u)$ represents the set $N(u) \cap X$.

A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Given $X \subseteq V(G)$, the *subgraph of G induced by X* is the graph $G[X] = (X, E_X)$, where $(u, v) \in E_X$ if and only if $u, v \in X$ and $(u, v) \in E(G)$. A *path* of G is any sequence $(v_1, \cdots, v_q)$, where $(v_i, v_{i+1}) \in E(G)$ for $i = 1, \ldots, q-1$ and $v_i \neq v_j$ for all $1 \leq i < j \leq q$; this path is a *cycle* of G if $(v_1, v_q) \in E(G)$. We denote such a path (or cycle) by $\langle v_1, \cdots, v_q \rangle$. If $P = \langle v_1, \cdots, v_q \rangle$ is a path, we call v_1, v_q the *extremities of P* and v_i is called an *internal vertex of P*, $i \in [2, q-1]$. The *length* of a path (or of a cycle) is the number of edges on the path (or cycle). An *induced path* is a path such that the only edges between the vertices of the path are the ones that define the path; an *induced cycle* is defined similarly.

If $u \in V(G)$ has degree at least $m(G) - 1$, we say that u is dense and we represent the set of all dense vertices of G by $D(G)$.

Let $G = (V, E)$ be any graph and ψ be a proper partial colouring of G. We denote the colour of a vertex $u \in V$ in ψ by $\psi(u)$ and the set of colours of the vertices in $X \subseteq V$ by $\psi(X)$. Now, let $W \subseteq V(G)$ of cardinality k such that $d(u) \geq k - 1$, for all $u \in W$, and let ψ be a proper partial colouring of G that uses k colours and is such that each vertex of W is coloured with a different colour in $\{1, \cdots, k\}$. For each $w \in W$, denote by $M_\psi(u)$ the set of colours $\{1, \cdots, k\} \setminus N[w]$ and by $U_\psi(w)$ the set of uncoloured neighbours of w (if there is no ambiguity, we use simply $M(w)$ and $U(w)$). If ψ is such that $|M_\psi(w)| \leq |U_\psi(w)|$, for every $w \in W$, we say that ψ is an *unsaturated precolouring of G with candidate set W*. Note that if all the vertices are coloured in an unsaturated precolouring with candidate set W, then all $u \in W$ is a b-vertex of ψ (this is true because there are no uncoloured vertices in the neighbourhood of a vertex $w \in W$ and, hence, $M_\psi(w) = 0$, for all $w \in W$). Thus, ψ is a b-colouring of G with basis W.

The following trivial lemma will be useful in some of the proofs.

Lemma 2. *Let G be a cactus and U and U' be two disjoint subsets of $V(G)$. If $G[U]$ and $G[U']$ are connected, then U has at most two neighbours in U'.*

3. Pivots and anomalous graphs

Let $W \subseteq D(G)$ with cardinality $m(G)$. In [1], Irving and Manlove define a vertex *encircled by W* as being a vertex $u \in V(G) \setminus W$ such that every $w \in W$ is either adjacent to u or to some $w' \in N(u) \cap W$ with degree $m(G) - 1$ (if w' exists, we say that w' is a *u, w-bridge*). Then, they prove the following important lemma:

Lemma 3 ([1]). *Let G be any graph and let $W \subseteq D(G)$ be a set of cardinality $m(G)$ that encircles $u \in V(G) \setminus W$ of degree less than $m(G)$. Then W is not a basis of a b-colouring of G with $m(G)$ colours.*

Proof: Suppose on the contrary that W is the basis of a b-colouring ψ of G with $m(G)$ colours. Let v be the vertex of W with the same colour as u. As

ψ is proper, we know that $v \notin N(u)$. So, let v' be the u, v-bridge. We get a contradiction as $d(v') = m(G) - 1$ and, therefore, all its neighbours should have different colours. $\square$

In this section, we present a family of cacti for which the difference between the b-chromatic number and the m-degree is at least 1. We know already, by Lemma 3, that if G is a cactus such that every subset of $D(G)$ with cardinality $m(G)$ encircles a vertex not in $D(G)$, then $\chi_b(G) < m(G)$. But also we define the following:

Let G be a cactus and W be a subset of $m(G)$ dense vertices of G. We say that W *encircles the pair* $x, y \in V$ if $x, y \notin W$, W encircles neither x nor y and one of the following occurs:

E1. There are $W' \subset W$ and $u, v \in W'$ such that $|W'| = m(G) - 1$, $\langle x, u, y, v\rangle$ is a cycle and:
  (a) $d(u) = d(v) = m(G) - 1$, $N^{W'}(u) \neq \emptyset$, $N^{W'}(v) \neq \emptyset$ and every $w \in W' \setminus \{u, v\}$ is adjacent to u or v; or
  (b) $d(u) = m(G) - 1$ and every $w \in W' \setminus \{u, v\}$ is adjacent to u; or
  (c) $d(u) = m(G)$, $d(v) = m(G) - 1$, $N^{W'}(u) \neq \emptyset$, $N^{W'}(v) \neq \emptyset$ and every $w \in W \setminus \{u, v\}$ is adjacent to u or v; or
  (d) $d(u) = m(G)$ and every $w \in W \setminus \{u, v\}$ is adjacent to u.

E2. There are $W' \subseteq W$ and $u, v, w \in W'$ such that $|W'| \geq m(G) - 1$, $\langle x, u, v, y, w\rangle$ is a cycle, $d(u) = d(v) = m(G) - 1$, every $w' \in W' \setminus \{u, v, w\}$ is adjacent to w, and either
  (a) $W' \subset W$ and $d(w) = m(G) - 1$; or
  (b) $W' = W$ and $d(w) = m(G)$.

Lemma 4. *Let G be a cactus and $W \subseteq D(G)$ be of cardinality $m(G)$. If W encircles a pair of vertices x, y, then W is not the basis of a b-colouring of G with $m(G)$ colours.*

Proof: Suppose on the contrary and let ψ be a b-colouring of G with W as basis. Consider that $\{1, \cdots, m(G)\}$ are the colours used in ψ and denote the vertex of W coloured with i by v_i, $i \in \{1, \cdots, m(G)\}$. Suppose first that E1 occurs and consider, without loss of generality, that $\langle v_1, x, v_2, y\rangle$ is a cycle in G. Suppose E1a or E1b occurs and let $W' = N^W[v_1] \cup N^W[v_2]$. If $\psi(x) = \psi(y)$, as at least one between v_1, v_2 has degree $m(G) - 1$, say v_1, we have that v_1 cannot be a b-vertex, a contradiction. So, consider that

$\psi(x) = j$, for some $v_j \in W'$ (recall that $|W \setminus W'| = 1$). Obviously, $j \notin \{1, 2\}$ and, hence, v_j is adjacent to either v_1 or v_2, say v_1. Observe that, in this case, $d(v_1) = m(G) - 1$ and we get a contradiction as the colour j appears twice in $N(v_1)$. Now, consider that E1c or E1d occurs and let $\psi(x) = i$ and $\psi(y) = j$. We know that $i, j \notin \{1, 2\}$ and v_i is either adjacent to v_1 or to v_2, the same being valid for v_j. Suppose, without loss of generality, that $d(v_1) = m(G)$. If E1c occurs, as $d(v_2) = m(G) - 1$ (i.e., we cannot repeat colours in $N(v_2)$), we have $v_i, v_j \in N(v_1)$; this also trivially holds when E1d occurs. We then get a contradiction as $r(v_1) \geq 2 > d(v_1) - m(G) + 1$. Finally, consider that E2 occurs and suppose, without loss of generality, that $\langle v_1, x, v_2, v_3, y \rangle$ is a cycle in G. One can verify, by analogous arguments, that v_1 cannot be a b-vertex in ψ as there are too many colours repeated in $N(v_1)$. $\square$

So, we know that if G is such that every subset of $D(G)$ with cardinality $m(G)$ either encircles a vertex or a pair of vertices, then $\chi_b(G) < m(G)$. Unfortunately, these definitions are not sufficient to describe all the cacti with $\chi_b(G) < m(G)$. Observe, for example, the graph G of Figure 1. We have $m(G) = 4$, the big vertices represent the dense vertices; if we colour $D(G)$ with $\{1, 2, 3, 4\}$ from left to right, we get that both u and v must be coloured 1 in order to turn the dense vertices of the cycle into b-vertices. Thus, G cannot be b-coloured with 4 colours. Now, consider G to be any graph in Figure 2 and let $H = G - \{(u, v)\}$ (remove only the edge (u, v)). We have $m(H) = m(G)$ and it is not hard to verify that any b-colouring of H with $m(H)$ colours is such that u and v have the same colour. Thus, G cannot be b-coloured with $m(G)$ colours. Actually, even if the black vertices in Figures 1 and 2 have degree bigger than $m(G) - 1$, the graph still cannot be b-coloured with $m(G)$ colours. We then say that a cactus G is *anomalous* if $G[D(G) \cup N(D(G))]$ contains a graph isomorphic to the graph in Figure 1 or to some graph in Figure 2, $|D(G)| = m(G)$ and $d(v) = m(G) - 1$, for every grey vertex in the figures.

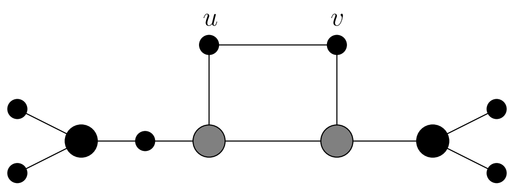

Figure 1: Anomalous graph with $m(G) = 4$.

We now analyse the possible number of encircled vertices and pairs of vertices of a subset W. The following propositions will be useful:

Proposition 5. *Let $W \subseteq D(G)$ with cardinality $m(G)$ and suppose W encircles $u \in V(G) \setminus W$ with degree less than $m(G)$. Then, $|N^W(u)| \geq 2$ and $|W \setminus N(u)| \geq 1$.*

Proof: Since $d(u) < m(G) = |W|$, there is a vertex w in $W \setminus N(u)$. Since W encircles u, there is a vertex $v \in N(w) \cap N(u) \cap W$ with degree $m(G)-1$. If v is the only neighbour of u in W, then, since W encircles u, all vertices of $W \setminus v$ must be adjacent to v; but then $d(v) \geq m(G)$, a contradiction. So u has at least two neighbours in W. □

Proposition 6. *Let W be any set of $m(G)$ dense vertices and let $u \in V \setminus W$ be encircled by W, or be one of the vertices of a pair encircled by W. Then, there are at least two vertices in W adjacent to u.*

Lemma 7. *Let W be a subset of $m(G)$ dense vertices. If W encircles two vertices, x and y, then one of the following occurs:*

- F1. *There are $u, v \in W$ such that $\langle x, u, y, v \rangle$ is a cycle, $d(u) = d(v) = m(G)-1$, $N^W(u) \neq \emptyset$, $N^W(v) \neq \emptyset$ and every $w \in W \setminus \{u, v\}$ is adjacent to u or v; or*
- F2. *There are $u, v, w \in W$ such that $\langle x, u, v, y, w \rangle$ is a cycle, $d(u) = d(v) = d(w) = m(G)-1$ and every $w' \in W \setminus \{u, v, w\}$ is adjacent to w.*
- F3. *$W = \{v_1, v_2, v_3, v_4\}$, $\langle x, v_1, v_2, y, v_3, v_4 \rangle$ is a cycle in G and $d(v_i) = 3$, $i = 1, \cdots, 4$.*

Proof: Note that $|N^W(x)| \leq 2$, otherwise, by Lemma 2, some neighbour of x would not be reached by y. So, by Proposition 6, we have $|N^W(x)| = 2$. Analogously, we have $|N^W(y)| = 2$. First, suppose x and y have at least one common neighbour in W, say w, and let $w_x \in N^W(x) \setminus \{w\}$ and $w_y \in N^W(y) \setminus \{w\}$. Observe that if $w_x = w_y$, then $W \nsubseteq N(w)$, otherwise we have $d(w) > m(G)-1$ and w cannot be a bridge. Analogously, $W \nsubseteq N(w_x)$ and, hence, $d(w) = d(w_x) = m(G)-1$ and (F1) occurs. Now, if $(w_x, w_y) \in E(G)$, then we must have $d(w_x) = d(w_y) = m(G)-1$ and $W \setminus \{w, w_x, w_y\} \subseteq N(w)$ (hence, $d(w) = m(G)-1$), i.e., (F2) occurs. So, suppose that $w_x \neq w_y$ and $(w_x, w_y) \notin E(G)$. Since $N^W(y) = \{w, w_y\}$, w must be the y, w_x-bridge; analogously, w is also the x, w_y-bridge. Observe Figure 3 and note that, in

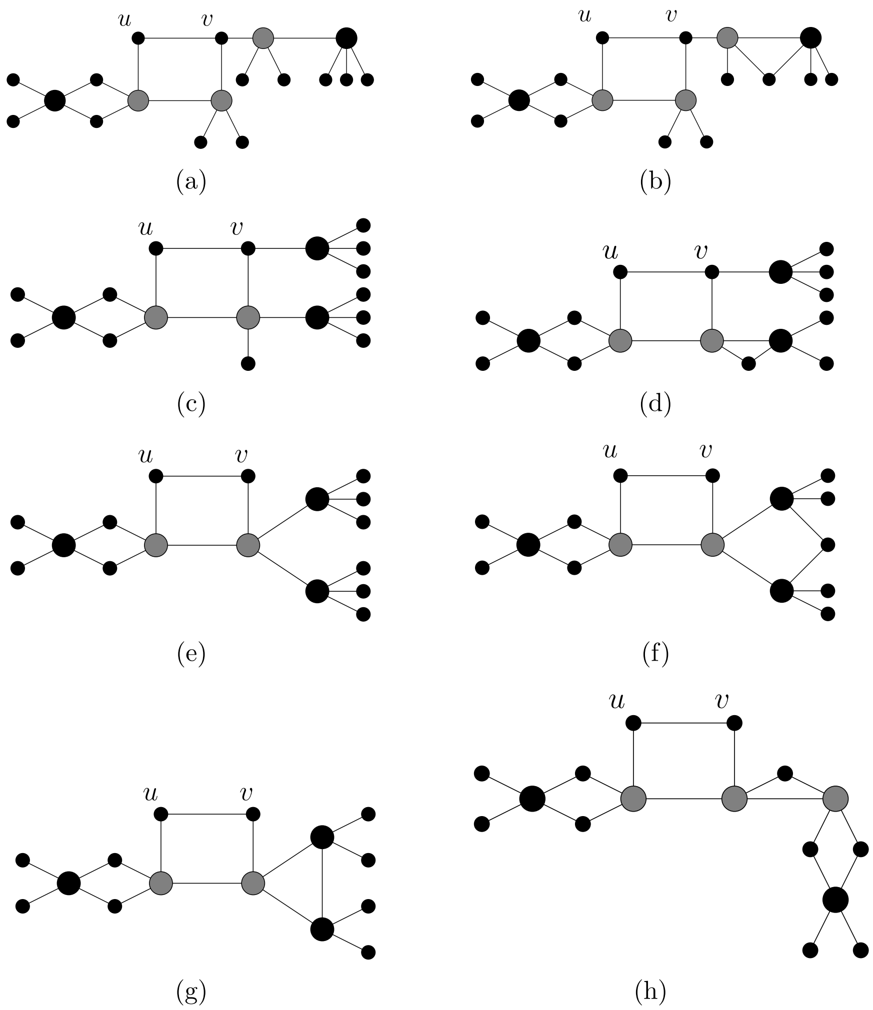

Figure 2: Anomalous graphs with $m(G) = 5$.

this case, w must also be a x, w'-bridge, for all $w' \in W \setminus \{w_x\}$, otherwise, w' is not reached by y. However, in this case, $(W \setminus \{w\}) \cup \{x, y\} \subseteq N(w)$, i.e., $d(w) \geq m(G) + 1$, a contradiction. Now, suppose x, y have no common neighbours in W and let $N^W(x) = \{u_x, v_x\}$ and $N^W(y) = \{u_y, v_y\}$. Note that, if both u_x and v_x have the same bridge to y, say u_y, then x does not reach v_y, a contradiction. So, they have different bridges and must be bridges themselves, i.e., F3 occurs for x, y.

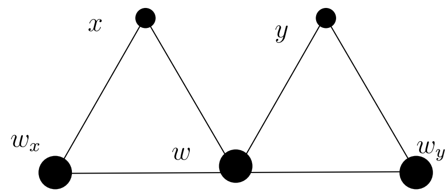

Figure 3: *Vertices x, y are encircled by W and have exactly one common neighbour in W.*

□

Given $W \subseteq D(G)$ and $x \in V(G) \setminus W$, we denote by W_x the set $W \cap (N(x) \cup N(N^W(x)))$. The following remark is trivially valid.

Remark 8. *If x, y is an encircled pair, then $W_x = W_y$ and $|W \setminus W_x| \leq 1$.*

Let $W \subseteq D(G)$ of cardinality $m(G)$. Observe that by the definition of encircled pair, W cannot encircle a pair of vertices and a vertex; also, by Lemma 7, we know that W encircles at most two distinct vertices. In the following lemma, we show the situations where W encircles more than one pair of vertices.

Lemma 9. *Let W be any set of $m(G)$ dense vertices, $m(G) \geq 4$. If W encircles more than one pair of vertices, then its structure is as represented in Figure 4, where the big vertices represent W and $d(v) = d(u') = m(G)-1$..*

Proof: Let x, y be an encircled pair of W. We analyse the existence of another encircled pair. Let x' be one of the vertices of such a pair, $x' \neq x, y$. First, suppose that E2 occurs for x, y and let $u, v, w \in W$ be such that $\langle x, w, y, v, u \rangle$ is a cycle in G. By Proposition 6 and Remark 8, x' must be adjacent to u or v. Suppose, without loss of generality, that $x' \in N(u)$ and let y' be the vertex forming an encircled pair with x'. Since G is a cactus and x', y' must be within a cycle of G, we have $y' \neq x, y$. By E2 and Remark 8, there exists

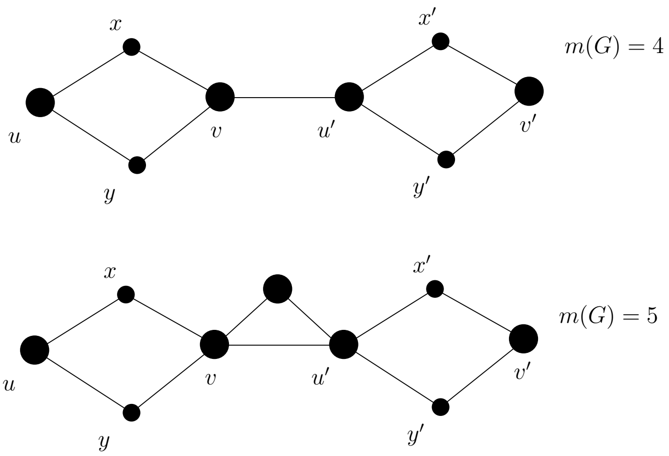

Figure 4: Cacti with two pairs encircled by W.

at most one vertex in $W \setminus \{u, v, w\}$ not adjacent to w; so, we must have the situation in Figure 5. Certainly, $W = \{u, v, w, w'\}$, otherwise there would be at least two different vertices in $W \setminus Z$, for $Z = W_x$ or $Z = W_{x'}$, contradicting Remark 8. However, in this case, $d(u) = 4 = m(G)$, contradicting E2.

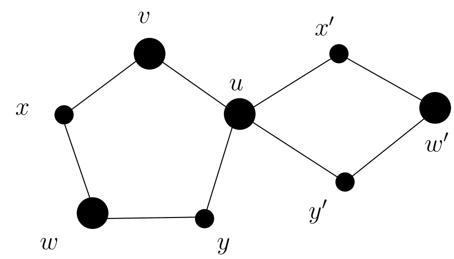

Figure 5: The pair x, y is encircled by W. Situation where E2 occurs for x and y and there exists another encircled pair $x', y,'$.

Now, suppose that E1 occurs for x, y and let $u, v \in W$ be such that $C = \langle x, u, y, v \rangle$ is a cycle in G. Let x', y' be an encircled pair different from x, y. By the paragraph above, we can suppose that E2 does not occur for x', y'. Furthermore, since two cycles may intersect in at most one vertex and by Remark 8 applied to the pairs x, y and x', y', we have $x' \neq x, y$ and $y' \neq x, y$. Let $C' = \langle x', u', y', v' \rangle$ be the cycle containing x', y', where $u', v' \in W$. Note that C, C' either intersect in one of the vertices u, v, u', v' or are connected through an edge between $\{u, v\}$ and $\{u', v'\}$. So, we can suppose that (1) $W \setminus (N[u'] \cup N[v']) = \{u\}$ and (2) $W \setminus (N[u] \cup N[v]) = \{v'\}$. Suppose C and C' intersect in vertex $v = u'$. Thus, $W \setminus \{u, v, v'\} \subseteq N(v)$ and $d(v) \geq m(G) + 1$, contradicting E1. Now, consider that the cycles are connected through the edge (v, u'). Note that every vertex w in $W^* = W \setminus \{u, v, u', v'\}$ must be adjacent to v and u'. So, $|W^*| \leq 1$ and the possible cases are the ones represented in Figure 4. Note that, by (1) and (2), we have that E1c and E1d do not occur; thus, $d(v) = d(u') = m(G) - 1$. □

4. Quasi-Good Set

In this section, we want to obtain a subset of the dense vertices of G that can play the role of a basis of a b-colouring of G with $m(G)$ colours. We say that $W \subseteq D(G)$ with cardinality $m(G)$ is a *quasi-good set* if (this definition is slightly different from the definition of "good set" in [1]):

- W does not encircle any vertex or pair of vertices; and
- Every $u \in V \setminus W$ with degree greater than $m(G) - 1$ is adjacent to some vertex in W.

Later on the text, we will use this quasi-good set to obtain a b-colouring of G with $m(G)$ colours, when $m(G) \geq 7$. If $|D(G)| = m(G)$, then: if $D(G)$ encircles a vertex or pair of vertices, then G does not have a quasi-good set; and if $D(G)$ does not encircle any vertex or pair of vertices, then $D(G)$ is a quasi-good set itself. So, it remains to analyse the existence of a quasi-good set in a cactus G that has more than $m(G)$ dense vertices. The main result of this section is the following:

Theorem 10. *If G is a cactus with $|D(G)| > m(G)$, then G does not have a quasi-good set if and only if $|D(G)| = m(G) + 1$ and either:*

(I) $D(G)$ is a cycle of length 5 and $d(v) = 3$, for all $v \in W$, or $D(G)$ is as represented in Figure 6, where $D(G)$ is represented by the bigger vertices and $d(v_2) = d(v_4) = 3$; or

(II) there exist vertices $u, v \in D(G)$, with degree $m(G) - 1$, and $w \notin D(G)$ such that $\langle u, v, w \rangle$ is a cycle and every vertex in $D(G)$ is adjacent to u or to v.

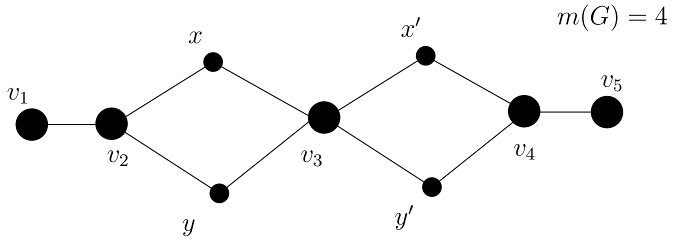

Figure 6: In this graph, $m(G) = 4$.

Note that if $|D(G)| = m(G) + 1$ and (I) or (II) occurs for $D(G)$, then G has no quasi-good set as every subset $W \subseteq D(G)$ with cardinality $m(G)$ either encircles a vertex or pair of vertices or is such that there exists $u \in V \setminus (W \cup N(W))$ with $d(u) > m(G) - 1$ (Figure 6). So, it remains to prove that this condition is sufficient, i.e., that if G does not have a quasi-good set, then $|D(G)| = m(G)+1$ and (I) or (II) occurs. Let $W \subseteq D(G)$ of cardinality $m(G) + 1$ that contains all vertices with degree at least $m(G)$. We actually prove that if one of the situations below occurs, then G has a quasi-good set:

1. $|D(G)| > m(G) + 1$ and (I) or (II) occurs for W; or
2. Neither (I) nor (II) occurs for W and some $W' \subseteq W$ with cardinality $m(G)$ encircles two vertices or a pair of vertices; or
3. Neither (I) nor (II) occurs for W and every $W' \subseteq W$ with cardinality $m(G)$ encircles at most one vertex and no pair of vertices.

Let W' be a set of $m(G)$ dense vertices containing all vertices with degree at least $m(G)$. If G does not have a quasi-good set, then W' encircles at least one vertex or a pair of vertices. Now, from 2 and 3, we get that (I) or (II) occurs and, from 1, we get that $|D(G)| = m(G)+1$. The theorem, then, follows. Now, we present lemmas that cover each described situation.

Recall that $W_x = W \cap (N(x) \cup N(N^W(x)))$. If $|W \setminus W_x| \geq 2$, we know that x is not encircled by W and, by Remark 8, that x is not part of a pair encircled by W. Also, obviously, if x is encircled by W, then $W_x = W$.

Lemma 11. *Let $W \subseteq D(G)$ of cardinality $m(G) + 1$ containing all vertices with degree at least $m(G)$. If $|D(G)| > m(G) + 1$ and (I) or (II) occurs for W, then G has a quasi-good set.*

Proof: Note that if the structure of W is as represented in Figure 6, then $(W \setminus \{v_2, v_4\}) \cup \{w\}$ is a quasi-good set, for any $w \in D(G) \setminus W$. So, we prove that if W is a cycle of length 5 or (II) occurs for W, then we can construct a quasi-good set.

Suppose that W is a cycle of length 5 and $d(v) = 3 = m(G) - 1$, for all $v \in W$. Thus, $d(v) = 3$, for all $v \in D(G)$ and if we get a subset $W' \subseteq D(G)$ of cardinality $m(G)$ that does not encircle any vertex or pair of vertices, then W' is a quasi-good set. Let v be any dense vertex not in W and suppose that $u \in W$ separates v from the cycle (if v is in another connected component, just consider any $u \in W$). Remove u and one of the vertices adjacent to u in W and add v, obtaining W'. Note that situations E1 and E2 cannot occur, since three of the vertices in W' form an induced path embedded in a cycle and $u \notin W'$ separates v from $W' \setminus \{v\}$. Furthermore, any vertex $w \in V \setminus W'$ does not reach v or at least one vertex of $W' \setminus \{v\}$. Thus, W' is a quasi-good set.

Now, suppose that (II) occurs and let v_1, v_2 be such that $d(v_i) = m(G) - 1$, $i = 1, 2$, $W \subseteq N(v_1) \cup N(v_2)$ and $\langle x, v_1, v_2 \rangle$ is a cycle, for $x \notin W$. Let W' be obtained from W by removing $\{v_1, v_2\}$ and adding any dense vertex in $D(G) \setminus W$. Since $d(v_1) = d(v_2) = m(G) - 1$ and W contains all vertices with degree at least $m(G)$, we have that, if W' does not encircle any vertex, or pair of vertices, then W' is a quasi-good set. Since $d(v_1) = m(G) - 1$ and $(W \setminus \{v_1\}) \cup \{x\}$ has cardinality $m(G) + 1$, there must exist at least two vertices in $W \setminus N[v_1]$, say v_3, v_4; hence, v_3, v_4 are adjacent to v_2. The same is valid for v_2; so, let v_5, v_6 be vertices of $W \cap (N(v_1) \setminus \{v_2\})$. Note that $v_3, v_4 \in W'$ are separated from $v_5, v_6 \in W'$ by v_1, v_2, where $v_1, v_2 \notin W'$. So, $|W \setminus W_{w'}| \geq 2$, for all $w' \in V \setminus W'$, and W' does not encircle any vertex or pair of vertices. $\square$

Lemma 12. *Let $W \subseteq D(G)$ of cardinality $m(G) + 1$ containing all vertices with degree at least $m(G)$. If neither (I) nor (II) occurs for W and some $W' \subseteq W$ with cardinality $m(G)$ encircles two vertices or at least one pair of vertices, then G has a quasi-good set.*

Proof: Let $W = \{v_1, \cdots, v_{m(G)+1}\}$ and denote by W^i the set $W \setminus \{v_i\}$. Suppose, without loss of generality, that W^1 encircles two vertices or at least one pair of vertices. First, consider the case where W^1 encircles more than one pair, i.e., the structure of G is as represented in Figure 4 (the big vertices represent W^1). Denote by S the set $W^1 \cup \{x, y, x', y'\}$. Since $d(v) = d(u') = m(G) - 1$, we know that $N(v), N(u') \subseteq S$. If v_1 is disconnected from u' in $G - v$, replace v by v_1, obtaining W'. Observe that $|W' \setminus W'_t| \geq 1$, for all $t \in V \setminus W'$, unless $t = v$. Thus, W' does not encircle any vertex and, by Remark 8, W' also does not encircle any pair of vertices. Consequently, as $d(v) = m(G) - 1$, W' is a quasi-good set. If v_1 is disconnected from v in $G - u'$, we have an analogous situation. Finally, if $m(G) = 5$ and v_1 is disconnected from $\{u', v\}$ in $G - z$, where $z \in N(v) \cap N(u')$, replace z by v_1, obtaining W'. Note that $|W' \setminus W'_t| \geq 2$, for all $t \in V \setminus W'$, and, as z is adjacent to $v \in W'$, we have that W' is a quasi-good set.

Now, suppose that W^1 either encircles two vertices or a pair of vertices. We analyse the possible cases, according to the definition of encircled pairs and to Lemma 7:

- E1 or F1 occurs for W^1: let x, y be two distinct vertices encircled by W^1 or a pair encircled by W^1 and suppose, without loss of generality, that v_2, v_3 are such that $C = \langle x, v_2, y, v_3 \rangle$ is a cycle in G. Let W' represent W^1_x. By E1 and F1, we can suppose that $N^{W'}(v_3) \neq \emptyset$. Furthermore, if E1a or E1c occurs, suppose that $d(v_2) = m(G) - 1$. Without loss of generality, suppose that $v_4 \in N(v_3)$ and, if E1a or E1b occurs, let v_t be the vertex in $W^1 \setminus W'$.

  Observe that (1) if $v_1 \notin \{x, y\}$ or $N^W(v_2) \setminus \{x, y\} \neq \emptyset$, then v_3 is not encircled by W^3. Moreover, no other vertex can be encircled by W^3. Thus, as v_3 is adjacent to $v_4 \in W^3$, if v_3 is not encircled by W^3 and W^3 encircles no pair of vertices, then W^3 is a quasi-good set.

  First, assume $v_1 = x$. Suppose that E1a, F1 or E1c occurs for W^1 and let $v_i \in N^{W^1}(v_2)$. By (1), v_3 is not encircled by W^3. Suppose that x', y' is a pair encircled by W^3. By Remark 8, both x' and y' must be adjacent to v_2 and $N^W(v_3) \setminus \{v_1, v_4\} = \emptyset$. So, $(W^3 \cup \{y, x', y'\}) \setminus \{v_2, v_4, v_t\} \subseteq N(v_2)$, which is a contradiction as $d(v_2) = m(G) - 1$. Now, suppose that E1b or E1d occurs. Note that, because of the degree of v_3, either $N(v_3) = (W \setminus \{v_2, v_3, v_t\}) \cup \{y\}$ or $N(v_3) = (W \setminus \{v_2, v_3\}) \cup \{y\}$, i.e., y is the only neighbour of v_3 not in W. Also, as neither x nor y are encircled vertices by the definition of encircled pair, we have that $v_t \notin N(y)$ and, thus, no vertex can be encircled by W^2. To see that W^2 is a quasi-good set, first note that v_2 is adjacent to $v_1 \in W^2$. By contradiction, suppose that W^2 encircles a pair and let C' be the cycle that contains this pair. Since $W^2 \setminus N[v_3]$ contains at most one vertex, we must have situation E1b for x, y and $v_t \in C'$. We know that C' contains at least one other vertex in W^2 and such a vertex can only be a neighbour of

v_3. Therefore, if W^2 encircles a pair, then note that W^1 encircles two pairs, a contradiction.

Now, assume $v_1 \neq x, y$. By (1), W^3 does not encircle v_3 and therefore encircles no vertex. So, suppose that W^3 encircles the pair x', y'. By Remark 8, we know that $x', y' \neq x, y$. Note that, as $|W^3 \setminus W^3_{x'}| \leq 1$, v_1 is disconnected from $\{x, y\}$ in either $G - v_2$ or $G - v_3$. Furthermore, if v_1 is disconnected from $\{x, y\}$ in $G - v_3$, then $N^W(v_2) = \emptyset$, and if v_1 is disconnected from $\{x, y\}$ in $G - v_2$, then $N^W(v_3) = \{v_4\}$. We analyse the following possibilities:

- v_1 is disconnected from $\{x, y\}$ in $G - v_3$: so, E1b or E1d occurs for W^1 and x, y. Let $D = \{v_2, v_3, v_t\}$, if situation E1b occurs, and let $D = \{v_2, v_3\}$, otherwise. Note that (i) $N(v_3) = (W^1 \setminus D) \cup \{x, y\}$. Suppose, first, that $x' = v_3$. By (i) and Proposition 6, we have that one of the situations in Figure 7 occurs. Note that W^4 does not encircle any vertex or pair of vertices and, as v_4 is adjacent to $v_3 \in W^4$, W^4 is a quasi-good set. Now, suppose that $x', y' \neq v_3$. By (i), we know that $x', y' \notin N(v_3)$. If $(N(x') \cup N(y')) \cap N(v_3) = \emptyset$, then we have the situation represented in Figure 8.(a) and one can check that W^i is a quasi-good set. Otherwise, let $v_i \in N(v_3) \cap (N(x') \cup N(y'))$. As $x', y' \notin N(v_3)$, $(x', y') \notin E(G)$, E1 or E2 occurs for W^3 and by (i), one can verify that the possible situations are illustrated in Figure 8.b,c,d and W^i is a quasi-good set.
- v_1 is disconnected from $\{x, y\}$ in $G - v_2$: so, $N^W(v_3) = \{v_4\}$ and (i) $W^3_{x'} = W^3_{y'} = W^3 \setminus \{v_4\}$. We have that v_i is also disconnected from $\{x, y\}$ by v_2, for all $v_i \in W^3 \setminus \{v_2, v_4\}$. Note that, if $N^{W^2}(v_2) \neq \emptyset$, then W^2 is a quasi-good set. Thus, suppose that $N^{W^2}(v_2) = \emptyset$; this implies $|W^1_x| = 3$ and therefore $m(G) = 4$. Also, $x', y' \in N(v_2)$. Note that E2 does not occur for x', y', by the existence of $v_4 \in W^3$ and the fact that $d(v_2) > m(G) - 1$. Thus, the structure of W is as in Figure 6, i.e., (I) occurs for W, a contradiction.

- E2 or F2 occurs for W^1: let x, y be two distinct vertices encircled by W^1 or a pair encircled by W^1. Suppose, without loss of generality, that $C = \langle x, v_2, v_3, y, v_4 \rangle$ is a cycle. By E2 and F2, we know that (i) $W^1 \setminus (N[v_4] \cup \{v_2, v_3\})$ has at most one vertex; if it is the case, let v_5 be such a vertex. First, suppose that $N^{W^1}(v_4) \neq \emptyset$. Note that W^4

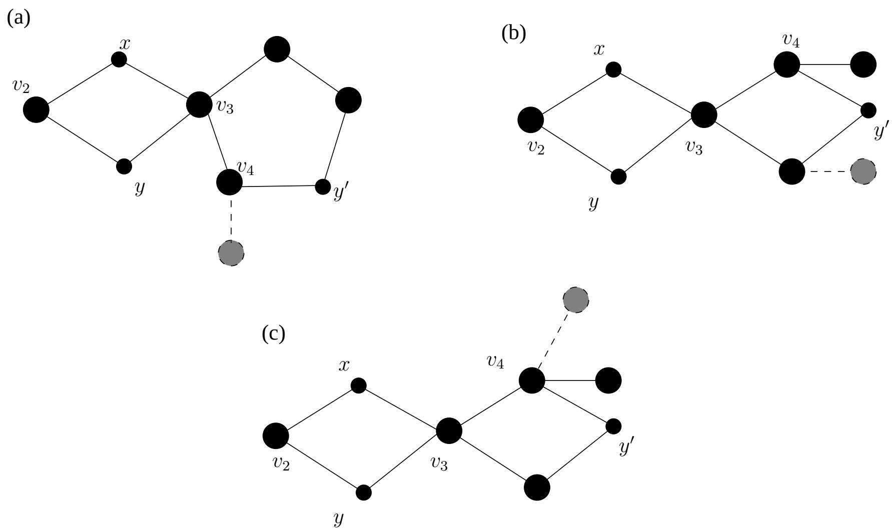

Figure 7: W^3 *encircles a pair when* v_3 *separates* v_1 *from* C. *The dotted edges and vertices may not exist.*

does not encircle any vertex and, if W^4 does not encircle any pair of vertices, we have that W^4 is a quasi-good set. So, suppose that W^4 encircles the pair x', y'. Note that x', y' are separated from C by either v_2 or v_3 (otherwise, $v_2, v_3 \notin W^4_{x'}$), say v_2. Therefore $|W^2 \setminus W^2_z| \geq 2$, for all $z \in V \setminus (W^2 \cup \{y\})$, and $|W^2 \setminus W^2_y| \geq 1$. Consequently, W^2 does not encircle any vertex or pair of vertices and, as $d(v_2) = m(G) - 1$, W^2 is a quasi-good set. Now, consider that $N^{W^1}(v_4) = \emptyset$. Since $m(G) \geq 4$, E2a must occur. Thus, $d(v_4) = m(G) - 1$ and if W^4 does not encircle any vertex or pair of vertices, then W^4 is a quasi-good set. Also, by (i), we have that $m(G) = 4$. First, suppose that W^4 encircles a vertex $z \in V \setminus W^4$. Note that if $z \in \{x, y\}$, then, as W^1 does not encircle z, we must have $d(v_1) = m(G) - 1$ and $v_1 \in N(v_5) \cap N(z)$. In this case, either W^2 is a quasi-good set, if $z = x$, or W^3 is a quasi-good set, if $z = y$. Now, consider that $z \neq x, y$; then, z must be adjacent either to v_2 or v_3, say v_2. As $N(v_2) = \{x, v_3, z\}$ and $W^4 = W^4_z$ (i.e., z separates C from v_1 and v_5), W^2 is a quasi-good set. Now, suppose that W^4 encircles a pair x', y'. As $d(v_2) = d(v_3) = 3$, neither v_2 nor v_3 are within the cycle containing x', y'; thus, the cycle containing x', y' contains v_1 and v_5. Also, by Remark 8, we have that $|W \setminus W^4_{x'}| \leq 1$ and either v_2 or v_3 is in $W \setminus W^4_{x'}$, say v_2. Thus, E1b occurs for W^4, i.e., one

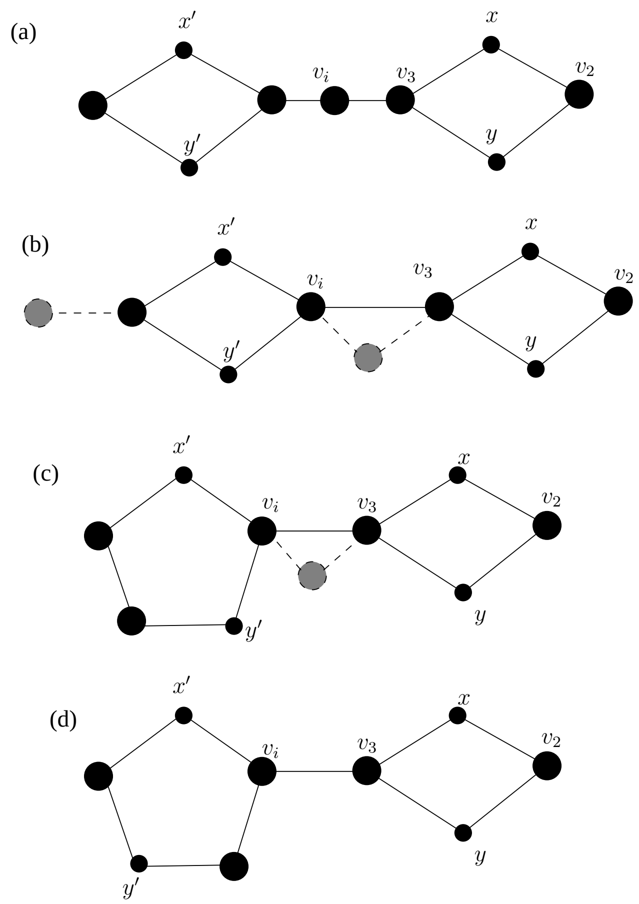

Figure 8: W^3 *encircles a pair when* v_3 *separates* v_1 *from* C. *The dotted edges and vertices may not exist, except for (b) where at least one of the dotted vertices must exist.*

between v_1 and v_5 is adjacent to v_3 and must have degree $m(G) - 1$, say v_1. In this case, W^3 is a quasi-good set (observe Figure 9).

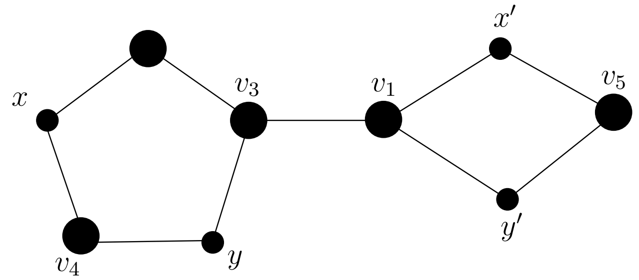

Figure 9: *E2 occurs for W^1, W^4 encircles the pair x', y' and $m(G) = 4$.*

- F3 occurs for W^1: let $\langle x, v_2, v_3, y, v_4, v_5 \rangle$ be the cycle, where x and y are encircled by W^1. Observe that if $x = v_1$, then W^3 does not encircle any vertex or pair of vertices and, as $d(v_3) = m(G) - 1$, we have that W^3 is a quasi-good set. If $v_1 = y$, we have an analogous situation. So, suppose that v_1 is not in the cycle. If v_1 is separated from the cycle by v_i, $i \in \{2, 3, 4, 5\}$, let $W' = W^i$; otherwise, if v_1 is separated from the cycle by x, let $W' = W^2$; finally, if v_1 is in another connected component of G or is separated from the cycle by y, let $W' = W^3$. Obviously, situations E1 and E2 cannot occur because of the disposition of three vertices of W' in the cycle of length six. Also, any vertex not in W' does not reach at least one vertex in W' and, since $d(v_j) = m(G) - 1$, for all $v_j \in W^1$, we have that W' is a quasi-good set.

$\square$

Lemma 13. *Let $W \subseteq D(G)$ of cardinality $m(G) + 1$ containing all vertices with degree at least $m(G)$. If neither (I) nor (II) occurs for W and every $W' \subseteq W$ with cardinality $m(G)$ encircles at most one vertex and no pair of vertices, then G has a quasi-good set.*

Proof: Consider the same notation as in the proof of the previous lemma. Suppose, without loss of generality, that $d(v_1) = m(G) - 1$ (note that there are at most $m(G)$ vertices with degree greater than $m(G) - 1$). Thus, if W^1 does not encircle any vertex, then it is a quasi-good set. So, let u be encircled by W^1 and suppose, without loss of generality, that $N(u) \cap W^1 = \{v_2, \cdots, v_{p-1}\}$.

By Proposition 5 and the fact that W^1 contains all vertices with degree at least $m(G)$, we know that $p > 3$ and $p-1 < m(G)+1$. We may also suppose that v_2 is a (u, v_p)-bridge; hence, $d(v_2) = m(G)-1$. We analyse the following cases:

- $u = v_1$: since $v_p \in N(v_2)$, if W^p does not encircle any vertex, then W^p is a quasi-good set. So, suppose that W^p encircles $v \in V \setminus W^p$. First, we analyse the case where v is adjacent to v_1. As $v_p \notin N(v_1)$, we have that $v \neq v_p$ and, consequently, $v \notin W$. By Proposition 5, there exists $v_i \in W^p$ adjacent to v, $i \neq 1$. We have that $N^{W^p}(v) = \{v_1, v_i\}$, otherwise v_1 does not reach some $v_j \in N^{W^1}(v)$. So, we know that every $v_j \in W^p \setminus \{v_1, v_i\}$ is either adjacent to v_1 or to v_i. Observe that, if $i = 2$, as $W \subseteq N(v_1) \cup N(v_2)$ and $d(v_1) = d(v_2) = m(G) - 1$, we have that (II) occurs for W, a contradiction; so, consider $i \neq 2$. If $N^{W^p}(v_i) \setminus \{v_1\} = \emptyset$, then $(W^p \setminus \{v_1\}) \cup \{v\} \subseteq N(v_1)$, a contradiction to the fact that $d(v_1) = m(G) - 1$. So, let $v_k \in N^{W^p}(v_i) \setminus \{v_1\}$. Note that any vertex in $V \setminus W^i$ does not reach at least one vertex in $\{v_3, v_p, v_k\}$, i.e., W^i does not encircle any vertex. Thus, as v_i is adjacent to $v_k \in W^i$, W^i is a quasi-good set.

  Now, suppose that v is not adjacent to v_1. Let v_i be a v, v_1-bridge (thus $d(v_i) = m(G) - 1$). By Proposition 5, there must exist $v_j \in W^p$ adjacent to v, $j \neq 1, i$. Also, v_j must be reachable from v_1 within W^1, i.e., either $v_j \in N(v_1)$ or there exists a v_1, v_j-bridge. Let $v_k = v_j$ if $v_j \in N(v_1)$, and let v_k be a v_1, v_j-bridge, otherwise. If $k = i$, then no neighbour of v_j in W^1 is reachable by v_1 and we get a contradiction as $(W^p \setminus \{v_i\}) \cup \{v\} \subseteq N(v_i)$ implies $d(v_i) > m(G) - 1$. So, suppose $k \neq i$ and let $C = \langle v_1, v_k, v_j, v, v_i \rangle$ be the cycle formed in G, $k = j$ or not. Note that $N^W(v) \setminus C = \emptyset$, since v_1 is encircled by W^1. Also, if there exists $v_l \in W^p \setminus C$, then W^l is a quasi-good set: since v_l must be reached by v within W^p and $N^{W^p}(v) \setminus C = \emptyset$, v_l must be adjacent to either v_i or v_j. Thus, consider $W = (C \setminus \{v\}) \cup \{v_p\}$. Since $|W| = m(G) + 1$ and $m(G) \geq 4$, we have that $k \neq j$. Consider, first, that $v \neq v_p$. Note that $k = 2$, otherwise W^1 encircles v and v_1. In this case, $W^i = N[v_2]$ (recall that $d(v_2) = m(G) - 1 = 3$) and, as v_i is adjacent to $v_1 \in W^i$, W^i is a quasi-good set. Now, consider that $v = v_p$; thus, $i = 2$. As (I) does not occur for W and $d(v_1) = d(v_2) = d(v_k) = d(v_j) = 3$, we must have that $d(v_p) > m(G) - 1$. In this case, W^2 is a quasi-good set.

- $u \neq v_1$ (hence, $u \notin W$: if $p > 4$, then v_2 is separated from at least one vertex $v_i \in W$ by u. In this case, we can verify that W^2 does not encircle any vertex and, as $d(v_2) = m(G) - 1$, W^2 is a quasi-good set. Now, suppose $p = 4$. We claim that, if there is no quasi-good set, then W^2 encircles v_2 and W^3 encircles v_3 (Claim 14). Suppose $(v_2, v_3) \notin E$. So, there must exist a v_2, v_3-bridge, v_i. By Proposition 6, there exists $v_k \in N^{W^2}(v_2) \setminus \{v_i\}$, contradicting the fact that v_3 is encircled by W^3. Thus, $(v_2, v_3) \in E$. Since $u \neq v_1$ is encircled by W^1 and $N^{W^1}(u) = \{v_2, v_3\}$, we know that every vertex in $W^1 \setminus \{v_2, v_3\}$ is either adjacent to v_2 or to v_3. In addition, since v_1 must be reachable from v_2 within W^2 and from v_3 within W^3 and by the existence of the cycle $\langle u, v_2, v_3 \rangle$, we know that v_1 must be adjacent either to v_2 or to v_3. Finally, as $d(v_2) = m(G) - 1$ and by Proposition 5, there exists $v_i \in W^2 \setminus N(v_2)$. As G is a cactus, v_3 must be the v_i, v-bridge (hence, $d(v_3) = m(G) - 1$). So, (II) occurs for W, a contradiction. To complete the proof of the lemma, we prove the following claim.

Claim 14. *Let u be encircled by W^1, $u \neq v_1$. If G has no quasi-good set, then W^2 encircles v_2 and W^3 encircles v_3.*

Proof: By contradiction, suppose that G has no quasi-good set and W^2 encircles v, $v \neq v_2$. First, consider $v = u$. Since $N^{W^1}(u) = \{v_2, v_3\}$ and by Proposition 5, we know that v_1 must be adjacent to u. Note that, if $|N^W(v_2)| > 1$, then u does not reach some $v_i \in W^2$, a contradiction. So, $N^W(v_2) = \{v_p\}$ and, as $m(G) \geq 4$ and $N^{W^1}(u) = \{v_2, v_3\}$, there must exist $v_i \in N^{W^1}(v_3)$. Also, v_3 must be the u, v_i-bridge in W^1; hence, $d(v_3) = m(G) - 1$. But then, since any vertex in $V \setminus W^3$ does not reach at least one between $\{v_1, v_2, v_p, v_i\}$, W^3 is a quasi-good set, a contradiction.

Now, suppose that $v \neq u$. If v_2 is reachable from v within W^1, we have that W^1 encircles two vertices, u and v, a contradiction. So, $v \notin N(v_2)$ and, if v_i is adjacent to both v and v_2, then either $i = 1$ or $d(v_i) \geq m(G)$. We analyse the following cases:

(a) $v_p \in N(v)$: so, $d(v_p) \geq m(G)$ and, since v is encircled by W^2 and $v_3 \in W^2$, we have the cycle $C = \langle u, v_2, v_p, v, v_i, v_3 \rangle$, where v_i is either v_3 or has degree $m(G)-1$ (and, thus, $v_i \neq v_p$). If $i \neq 3$, note that any $w \in V \setminus W^3$ is distant from at least one between $v_2, v_p, v_i \in$

W^3, so W^3 does not encircle any vertex and, as v_3 is adjacent to $v_i \in W^3$, W^3 is a quasi-good set, a contradiction. So, consider $i = 3$. As $m(G) \geq 4$, there must exist $v_i \in W^2 \setminus \{v_1, v_3, v_p\}$ and, since v_i is also in W^1, v_3 must be a (u, v_i)-bridge and a (v, v_i)-bridge (thus, $d(v_3) = m(G) - 1$). Note that any vertex $w \in V \setminus W^3$ does not reach at least one between v_2, v_p, v_i and W^3 is a quasi-good set, a contradiction.

(b) $v_p \notin N(v)$: since $v_p \in W^2$ and v is encircled by W^2, there must exist a v, v_p-bridge, v_i. First, we analyse the case where $v_i = v_3$ or is a v, v_3-bridge; thus, there exists a cycle $\langle u, v_3, v_i, v_p, v_2 \rangle$. By Proposition 5, there exists $v_k \in N(v) \setminus \{v_i\}$. Observe Figure 10. If $i \neq 3$, note that $k = 1$, as v_k is not reachable by u. So, $m(G) = 4$ and, since $d(v_i) = m(G) - 1$, we have that $N(v_i) = \{v, v_3, v_p\}$. However, in this case, any $w \in V \setminus W^i$ does not reach at least one between v_1, v_2, v_3, v_p and, consequently, W^i is a quasi-good set, a contradiction. So, consider $i = 3$. Note that, there is no $v_l \in N^{W^1}(v_2) \setminus \{v_p\}$, otherwise v_l is distant from v within W^2. Since $N^{W^1}(u) = \{v_2, v_3\}$, we have that all vertices in $W^1 \setminus \{v_2, v_3\}$ must be adjacent to v_3. However, in this case, $N(v_3)$ contains the set $(W^1 \setminus \{v_2, v_3\}) \cup \{u, v\}$ with cardinality $m(G)$, contradicting the fact that v_3 is a v, v_p-bridge.

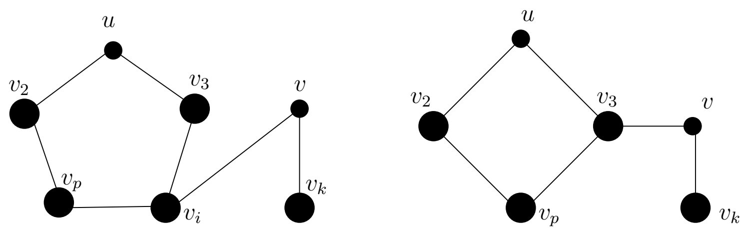

Figure 10: Situation of case (b) of the proof of Claim 14, where the v, v_p-bridge is within a cycle containing u, but not v.

Now, suppose that the v, v_p-bridge, v_i, is not in $N[v_3]$. Since v_2 is not reachable from v within W^1, we have that v_i cannot be adjacent to v_2 and, hence, v_i is not reachable from u within W^1, i.e., $i = 1$ (recall that $N^{W^1}(u) = \{v_2, v_3\}$). Also, as v_3 is reachable from v, we have that $\langle u, v_2, v_p, v_1, v, v_j, v_3 \rangle$ is a cycle in G, where $j = 3$ or v_j is a v, v_3-bridge. Observe Figure 11. Note that any vertex $w \in V \setminus W^p$ does not reach at least one between v_1, v_2, v_3, i.e., W^p does not encircle any vertex and, as v_p is adjacent to $v_2 \in W^p$, W^p is a quasi-good set, a contradiction.

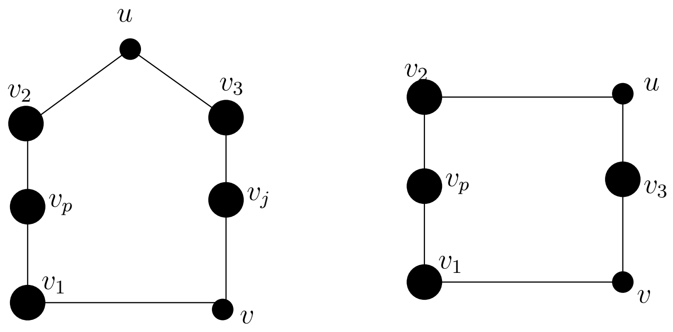

Figure 11: Situation of case (b) of the proof of Claim 14, where the v, v_p-bridge is within a cycle containing u and v.

This completes the proof that W^2 encircles v_2. Now, observe that if $d(v_3) = m(G) - 1$ and $N^{W^1}(v_3) \neq \emptyset$, then we can prove that v_3 is encircled by W^3 analogously. So, suppose otherwise. As $N^{W^1}(u) = \{v_2, v_3\}$, v_2 must be a u, v_i-bridge, for all $v_i \in W^1 \setminus \{v_2, v_3\}$. In fact, $N(v_2) = (W^1 \setminus \{v_2, v_3\}) \cup \{u\}$, i.e., $v_3 \notin N(v_2)$. So, if v_2 is encircled by W^2, there must exist $v_i \in N^{W^2}(v_2) \cap N(v_3)$ such that $d(v_i) = m(G) - 1$. Also, as $m(G) \geq 4$, we have that there exists $v_j \in W^1 \setminus \{v_2, v_3, v_i\}$ which implies that W^3 does not encircle any vertex. Therefore, as v_3 is adjacent to $v_i \in W^3$, we have that W^3 is a quasi-good set, a contradiction.

■

□

5. b-colouring anomalous cacti and cacti with no quasi-good set

In this section, we colour anomalous cacti and cacti that have no quasi-good set. Let G be such a cactus and let $W \subseteq D(G)$ of cardinality $m(G) - 1$. We say that a path P between $u, v \in W$ is a *link in* W if P has length at most three and every internal vertex of P is not in W; if $u \notin W$ is in a link, we say that u is a *link vertex of* W. Denote the set of link vertices of W by L. Now, let ψ be an unsaturated precolouring with candidate set W that colours $D(G) \cup L$; then, ψ can be easily extended to a b-colouring of G with $m(G) - 1$ colours: first, turn each $w \in W$ into a b-vertex (can be done separatedly and independently, as L is coloured); then, colour each $u \in V$ still uncoloured with any colour not in $N(u)$ (such a colour exists as u has degree at most $m(G) - 2$). So, in the remaining of this section, we will construct such a precolouring. The following proposition will be useful in some of the upcoming subsections:

Proposition 15. *Let $W \subseteq D(G)$ with cardinality $m(G)$ and let $v \in W$ be such that $d(v) = m(G) - 1$. Then, $|W \setminus N[v]| = |N(v) \setminus W|$.*

Proof: Denote by q the value $|N(v) \setminus W|$. We have that $|N(v)| = |N(v) \setminus W| + |N^W(v)| \Rightarrow m(G) - 1 = q + |N^W(v)|$. So, $q = m(G) - |N^W(v)| - 1$. Also, we know that $|W \setminus N[v]| = |W| - |N^W(v)| - 1$ and the result follows. □

Observe first that, for anomalous cacti, the precolourings presented in Figure 12 are as desired, where W is represented by the grey vertices. Also, if $|D(G)| = m(G)$ and $D(G)$ encircles two pairs of vertices, by Lemma 9, G is as represented in Figure 4; then, the precolouring ψ presented in Figure 13 is as desired, where W is represented by the grey vertices. In the upcoming subsections, we analyse the remaining cases where G does not have a quasi-good set.

5.1. $|D(G)| = m(G)$ and $D(G)$ encircles a pair of vertices

Suppose that $|D(G)| = m(G)$ and $D(G)$ encircles exactly one pair of vertices. In each possible situation where this happens, we choose a subset $W \subset D(G)$ of cardinality $m(G) - 1$ and construct an unsaturated precolouring with candidate set W. Note that, as a dense vertex has degree at least $m(G) - 1$, if we ensure that at most one neighbour of v has a repeated colour, for each $v \in W$, then the obtained precolouring is unsaturated. We need the following remark:

Remark 16. *Let $a, b \in W$. There are at most three links between a and b, not necessarily disjoint, and at most two neighbours of a and two neighbours of b lie in these paths (observe Figure 14).*

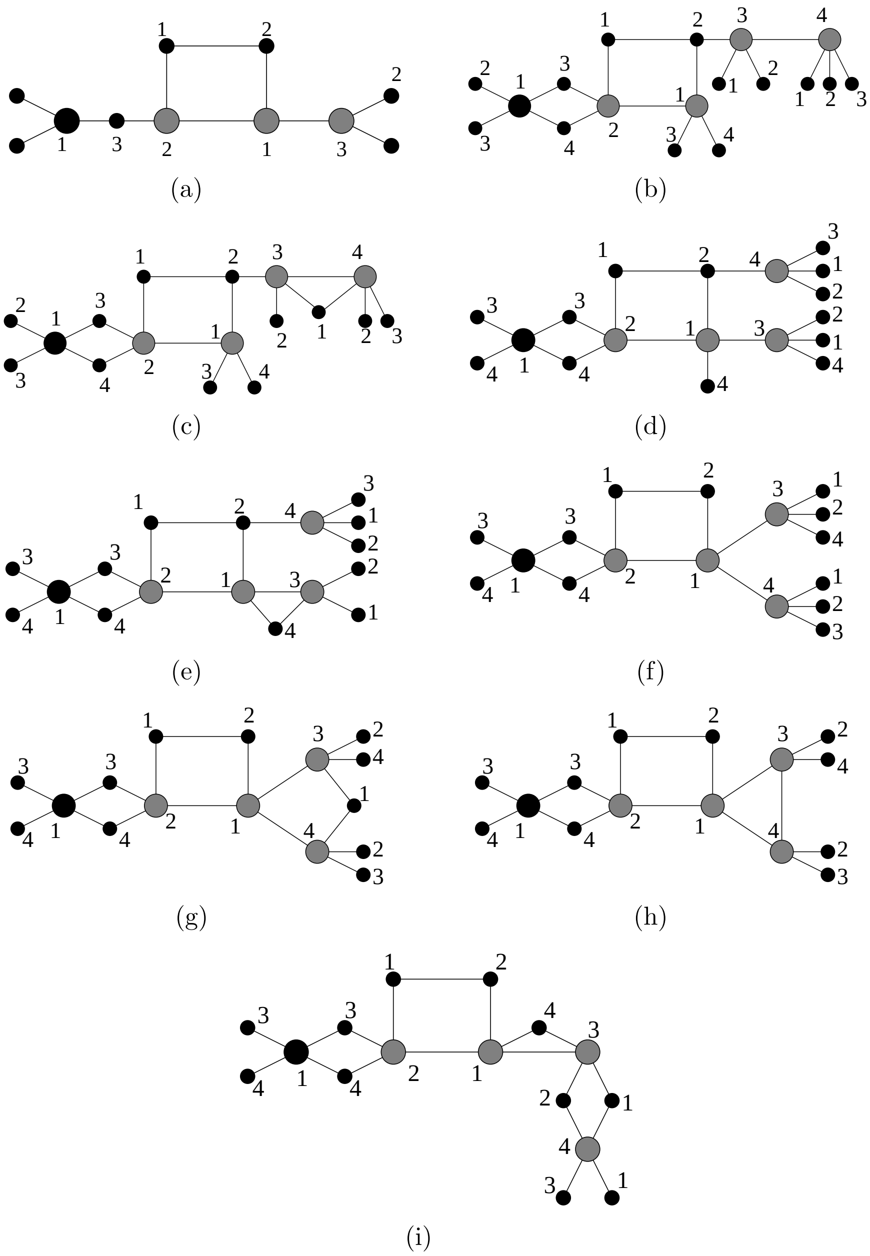

Figure 12: Precolouring of anomalous cacti.

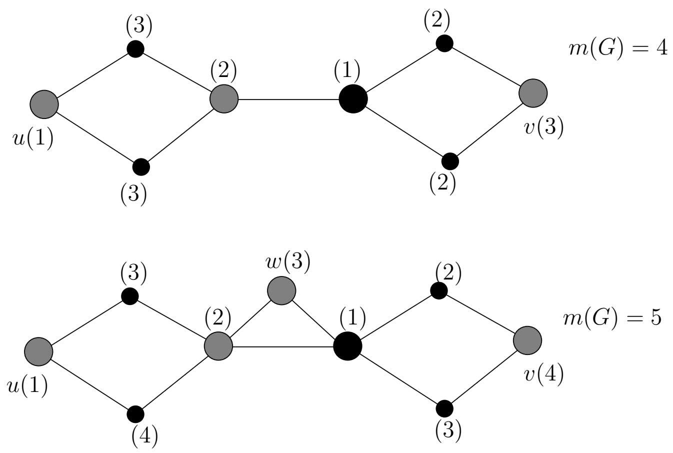

Figure 13: Precolouring of a graph with structure as represented in Figure 4.

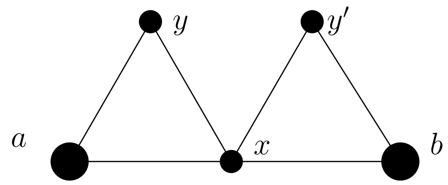

Figure 14: Situation when there are three link between $a, b \in W$: $\langle a, x, b\rangle$, $\langle a, y, x, b\rangle$ and $\langle a, x, y', b\rangle$.

Suppose that $D(G)$ encircles the pair x, y and let $D(G) = \{v_1, \cdots, v_{m(G)}\}$. By definition, we have one of the following cases:

- E1 occurs: suppose, without loss of generality, that $\langle x, v_1, y, v_2 \rangle$ is a cycle in G. Also, as x and y are not encircled vertices, we can suppose that $v_{m(G)}$ is not reachable from x or from y. Let $W' = D(G) \setminus \{v_{m(G)}\}$, if E1a or E1b occurs, or $W' = D(G)$, otherwise. Finally, we can suppose that $N^{W'}(v_1) \neq \emptyset$ and that, if E1a or E1c occurs, then $d(v_2) = m(G) - 1$.

  Now, let $W = D(G) \setminus \{v_1\}$ and suppose, without loss of generality, that $v_3 \in N(v_1)$. Apply colour i to v_i, for all $1 < i < m(G)$, colour 1 to $y, v_{m(G)}$, colour 3 to x and colour 2 to v_1. Since the edges (v_1, v_2), (x, v_3) and $(y, v_{m(G)})$ are not in the graph, we have a proper precolouring. Furthermore, note that at most one vertex of W has a repeated colour in its neighbourhood, namely v_2, in the case $w_{m(G)} \in N(v_2)$. Thus, the obtained precolouring ψ is unsaturated. Now, we need to colour the remaining link vertices. Remark 17 follows directly from Remark 16 and the fact that x, y is a pair encircled by $D(G)$.

  Remark 17. *Vertex $v_{m(G)}$ has at most two link neighbours and, if $v_{m(G)} \in N(v_k)$, for some $v_k \in D(G)$, then $v_{m(G)}$ has at most one link neighbour different from v_1.*

  By Remark 17, no matter which colour we apply to the link neighbours of $v_{m(G)}$, as long as the partial colouring remains proper, we obtain an unsaturated extension of ψ. So, from now on, we will only be concerned about not repeating too many colours in $N(v_i)$, for $1 < i < m(G)$. We analyse the existence of the following types of links (they are coloured in the order presented below):

  - Links with extremity in v_2: let $x_1, \cdots, x_q$ be all the link neighbours of v_2 and denote by v_{i_j} the other extremity of the link passing through x_j, $j = 1, \cdots, q$. First, suppose that $d(v_2) = m(G) - 1$. Thus, as $\{x_1, \cdots, x_q, x, y\} \subseteq N(v_2) \setminus D(G)$, by Proposition 15, we have that $|D(G) \setminus N[v_2]| \geq q + 2$. Consequently, there exist at least q vertices in $D(G) \setminus \{v_1, v_2, v_{m(G)}\}$ non-adjacent to v_2 (and, hence, adjacent to v_1) and we can give the colours of these vertices to $x_1, \cdots, x_q$. Now, suppose that $d(v_2) \geq m(G)$; then, we know that E1a and E1c do not occur and, consequently, $W' \setminus \{v_1, v_2\} \subseteq$

$N(v_1)$. Also, observe that $q \leq 2$ and $i_j = m(G)$, for all $j \in \{1, q\}$. So, we can colour x_1, x_q with colours from $M(v_2)$ (if there are no such colours, just repeat colour 3 in x_1, x_q). At the end, if there is z in a link between v_2 and some $v_i \in W$, colour it with either 2 or 3.

- Links between v_i and v_j, $i, j \in \{3, m(G) - 1\}$: let $\langle v_i, x', y', v_j \rangle$ be such a link. If $x' \neq y'$, give colour j to x', if x' is not coloured yet, and colour i to y', if y' is not coloured yet; otherwise, give colour 3 to x'. Note that if some $v_k \in N(v_2)$ is the extremity of a link of this type, then there is no link between v_2 and v_k (i.e., v_2 is the only coloured neigbour of v_k until now). So, this step does not repeat colours in $N(w)$, for all $w \in W$.
- Links between v_i and $v_{m(G)}$, $i \in \{3, \cdots, m(G) - 1\}$: let z, z' be all the uncoloured link neighbours of v_i within a link with extremity in $v_{m(G)}$ (if there is only one such vertex, consider $z' = z$). Note that v_i has at most two coloured neighbours different from $v_{m(G)}$, namely v_j, where $j \in \{1, 2\}$, and some eventual x' in a path between v_i and some $v_k \in N[v_j]$, $k \in \{2, \cdots, m(G) - 1\}$. So, if $m(G) \geq 7$, there exist at least two colours not in $\{1, i, j, k\}$ that we can use to colour z and z'. Now, consider $m(G) \leq 6$. Suppose, first, that $v_i \in N(v_1)$ and that v_i has another coloured neighbour $x' \neq v_1, v_{m(G)}$. Since $v_1 \notin W$, we know that there exists a path of length at most three $\langle v_i, x', y', v_j \rangle$, where $v_j \in N(v_1)$. If $x' \neq v_j$, give colour 1 to x' and colour j to z and z'. Otherwise, suppose that $N(v_i) = \{v_1, v_j, z, z'\}$ and $z, z' \in N(v_{m(G)})$. One can verify that, in this case, $D(G)$ encircles two pairs, z, z' and x, y, a contradiction. So, there must exist a neighbour of v_i non-adjacent to $v_{m(G)}$ that we can colour with 1; so, colour z, z' with $c \in \{1, \cdots, m(G) - 1\} \setminus \{1, 2, i, j\}$ (if $m(G) = 5$, just repeat colour j in z and z'). Now, consider that v_1 is the only coloured neighbour of v_i. If $m(G) > 4$, there must exist a neighbour of v_i non-adjacent to $v_{m(G)}$ that we can colour with 1; then, we colour z and z' with the colour (or colours) in $\{1, \cdots, m(G) - 1\} \setminus \{1, 2, i\}$. Otherwise, if $m(G) = 4$ (thus, $i = 3$), suppose that $N(v_3) = \{x', y', v_1\}$ and $x', y' \in N(v_4)$. Note that, as $D(G) \cap N(v_2) = \emptyset$ and $v_4 \notin N(v_1)$, E1b must occur and $d(v_1) = 3$. However, in this case, $D(G)$ encircles two pairs, x', y' and x, y, a contradiction. So,

there must exist a neighbour x' of v_3 non-adjacent to v_4; colour x' with 1 and z, z' different from x' with 2 (one can verify that $z, z' \notin N(v_1)$ as $|D(G)| = 4$). Finally, suppose that $v_i \in N(v_2)$. If $v_{m(G)} \in N(v_2)$, we colour the link between v_i and $v_{m(G)}$ as in the previous item; so, consider $v_{m(G)} \notin N(v_2)$. Note that $m(G) = 5$, $i = 4$, $N^{D(G)}(v_2) = \{v_4\}$ and $N^{D(G)}(v_1) = \{v_3\}$. Colour z, z' with 3 and, if there exists a link $\langle v_2, x', y', v_4\rangle$, change the colour of y' to 1. If, at the end, $v_{m(G)}$ has a link neighbour y' still uncoloured, then give colour i to y', where v_i is the other extremity of the link passing by y'.

- E2 occurs: suppose, without loss of generality, that $\langle x, v_1, v_2, y, v_{m(G)}\rangle$ is a cycle in G and that, if E2a occurs, then $v_{m(G)-1} \in D(G) \setminus N(v_{m(G)})$. Let $W = D(G)\setminus\{v_{m(G)}\}$. Colour v_i with i, for all $i \in \{1, \cdots, m(G)-1\}$, x and y with $m(G)-1$ and $v_{m(G)}$ with 1. Note that, as neither x nor y is encircled by $D(G)$, $v_{m(G)-1} \notin N(z)$, for all $z \in \{x, y, v_1, v_2\}$; thus, the precolouring is proper and unsaturated. Now, we need to colour the link vertices of W. First, consider a link $\langle v_i, x', y', v_j\rangle$, where $v_i, v_j \in N(v_{m(G)})$: if $x' \neq y'$, give colour i to y' and colour j to x'; otherwise, give colour 2 to x'. Now, if there is still some uncoloured link vertex, note that such a vertex lies within a link with extremity in $v_{m(G)-1}$ and $v_{m(G)-1} \notin N(v_{m(G)})$ (hence, E2a occurs). So, let z, z' be all the uncoloured link neighbours of v_i, $i \in \{1, \cdots, m(G)-2\}$. We know that no colour has been repeated in $N(v_i)$; thus, if there exists $c \in M(v_i) \setminus \{m(G)-1\}$, then we can colour z and z' with c. We prove that this colour exists. As v_1 and v_2 have exactly two coloured neighbours and $m(G) \geq 5$ ($D(G)$ contains at least two vertices not in $\{v_1, v_2, v_{m(G)}\}$, namely v_i and $v_{m(G)-1}$), we know that $M(v_j) \setminus \{m(G)-1\} \neq \emptyset$, for $j \in \{1, 2\}$. So, consider $v_i \in N(v_{m(G)})$. By an analogous argument, we can suppose that v_i has more than two coloured neighbours. The only situation where it happens is when $v_{m(G)-1} \in N(v_i)$ and there exists a path of length at most 3, $\langle v_i, x', y', v_j\rangle$, for some $j \in \{3, \cdots, m(G)-1\}$; thus, $m(G) \geq 6$ and, trivially, $\{1, \cdots, m(G)-2\} \setminus \psi(N[v_i]) \neq \emptyset$, i.e., colour c exists. We can apply an analogous argument to colour any uncoloured link neighbour of $v_{m(G)-1}$ at the end. We then obtain an unsaturated precolouring ψ with candidate set W that colours $D(G)$ and all link vertices of W.

5.2. $|D(G)| = m(G)$ *and* $D(G)$ *encircles a vertex* u

Let $D(G) = \{v_1, \cdots, v_{m(G)}\}$ and suppose that $D(G)$ encircles $u \in V \setminus D(G)$. By Lemma 5, we can suppose, without loss of generality, that $v_1 \notin N(u)$ and that $v = v_{m(G)}$ is a u, v_1-bridge; also, there must exist $v_q \in N(u) \setminus \{v\}$ (if there exists v_q that is within a cycle with v, then choose this vertex).

Let $W = D(G) \setminus \{v\}$. Apply colour i to v_i, $1 \leq i < m(G)$, colour 1 to u and colour q to v. Note that the only situations where we repeat colours in the neighbourhood of some vertex of W are: if there exists $v_k \in N(u) \cap N(v_1)$, in which case, by the choice of v_q, we have $k = q$; or if there exists $v_k \in N(v) \cap N(v_q)$, in which case we can suppose that $k = 1$. So, we can suppose that

(*) at most two vertices of W have repeated colours in their neighbourhoods, namely v_1 and v_q. Furthermore, if they do have repeated colours in their neighbourhoods, then $\langle u, v, v_1, v_q \rangle$ is a cycle in G.

Now, consider $v_i \in N^W(u)$ and let $S \subseteq N^W(v_i)$ be the neighbours of v_i having a link with v_i. Let $\{x_1, \cdots, x_r\} \subseteq N(v_i)$ be such that $x_j \in N(u) \cap N(v_i)$ or x_j is within a link between v_i and some $v_{i_j} \in S$. Note that, if $v_j \in S$, then $v_j \notin N(u)$, the link between v_i and v_j is the only one with extremity in v_j and $N^W(v_j) = \{v_i\}$. Also, if $x_j \in N(u) \cap N(v_i)$, then v_i is the only neighbour of x_j in W. Now, if $r = 1$ and $x_1 \in N(u)$, then we colour x_1 with any colour in $M(v_i)$ (if $M(v_i) = \emptyset$, colour x_1 with any colour different from $1, i$). Otherwise, we have $S \neq \emptyset$ and, consequently, $d(v_i) = m(G) - 1$ (v_i must be a u, v_j-bridge, for all $v_j \in S$). By Proposition 15, since $\{x_1, \cdots, x_r, u\} \subseteq N(v_i) \setminus W$, we know that there must exist at least r vertices in $W \setminus \{v_1\}$ non-adjacent to v_i whose colours we can use to colour $x_1, \cdots, x_r$. By what was said before, this step does not increase the number of repeated colour in $N(w)$, for all $w \in W$, and (*) still holds.

Now, we colour the remaining link vertices. We claim that if $v_i \in W$ has an uncoloured link neighbour x, then x is the only uncoloured link neighbour of v_i and there is no repeated colour in $N(v_i)$. Note that if this holds, then we can just colour x with any colour in $\{1, \cdots, m(G) - 1\} \setminus \psi(N(x))$ (as $x \notin D(G)$, we have that $d(x) < m(G) - 1$ and such a colour must exist). So, suppose that there exists a link $P = \langle v_i, x, y, v_j \rangle$ where x is uncoloured. If $(v_i, v_j) \in E(G)$, then we know that $x \neq y$ and $v_j \in N(u)$, as otherwise x would have been coloured in the previous paragraph. As pointed out before, this is the only link with extremity in v_i; also, there is no cycle $\langle u, v_q, v_1, v \rangle$

containing v_i and, by (*), we know that no colour in repeated in $N(v_i)$. Now, suppose that $(v_i, v_j) \notin E(G)$. As u is encircled by $D(G)$, there exists a cycle C containing the link P and either u or some $v_k \in N^{D(G)}(u)$. In either case, P is the only link of this type with extremity in v_i, i.e., x is the only uncoloured link neighbour of v_i. Also, as $(v_i, v_j) \notin E$ and by (*) and the existence of the cycle C, we know that there is no repeated colours in $N(v_i)$.

5.3. Cacti with $m(G)+1$ dense vertices and no quasi-good set

Now, we colour the graphs that, although having more than $m(G)$ dense vertices, do not have a quasi-good set. By Theorem 10, we know that $|D(G)| = m(G) + 1$ and one of the following situations occurs:

- The structure of $D(G)$ is as represented in Figure 6: we know that $m(G) = 4$, $d(v_1) = d(v_2) = 3$ and the link vertices of $D(G) \setminus \{v_1, v_2\}$ are $\{v_1, v_2, x, y, x', y'\}$. The precolouring presented in Figure 15 is as desired, where the grey vertices represent W.

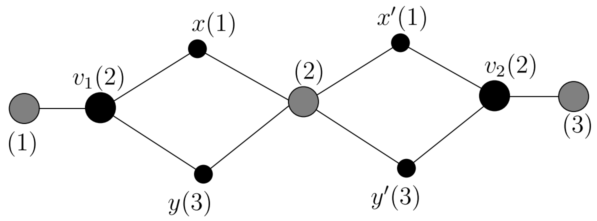

Figure 15: Precolouring of a graph whose structure is as in Figure 6.

- $D(G)$ induces a cycle of lenght 5 and $d(u) = 3$, for all $u \in D(G)$: let $\langle v_1, v_2, v_3, v_4, v_5 \rangle$ be the cycle induced by $D(G)$. Colour v_i with i, for $i \in \{1, 2, 3\}$, v_4 with 1 and v_5 with 3. The obtained precolouring is unsaturated with candidate set $W = \{v_1, v_2, v_3\}$ and colours $D(G)$ and all link vertices of W.

- There are vertices u, v with degree $m(G) - 1$ and a non-dense vertex, x, such that $\langle u, v, x \rangle$ is a cycle in G and $D(G) \subseteq N(u) \cup N(v)$: let $W = D(G) \setminus \{u, v\} = \{v_1, \cdots, v_{m(G)-1}\}$ and colour each $v_i \in W$ with colour i. Since $d(u) = d(v) = m(G) - 1$ and $x \in N(u) \cap N(v)$, there must exist $v_i \in N^W(u)$ and $v_j \in N^W(v)$. Colour u with j and v with i. Now, suppose that there is an uncoloured link $\langle v_a, x, y, v_b \rangle$. If $x \neq y$, then colour x with b and y with a. If $x = y$, then: if $v_a, v_b \in N(u)$, colour x with colour j; otherwise, colour x with colour i. Note that each $v_k \in N^W(u)$ has at most two coloured neighbours, namely u and some x within a path between v_k and some $v_j \in N^W(u)$. The same is analogously valid for the vertices in $N^W(v)$. Then, clearly, there is at most one vertex with a repeated colour in $N(v_i)$, for all $v_i \in W$, and the obtained precolouring is an unsaturated precolouring with candidate set W that colours $D(G)$ and all link vertices of W, as desired.

6. Colouring non-pivoted cacti

The main result of this section is the following.

Theorem 18. *Let G be a cactus with $m(G) \geq 7$ and W be a quasi-good set of G. Then, there exists a b-colouring of G with basis W.*

Let G be a cactus with $m(G) \geq 7$. Given a quasi-good set W of G, we will construct an unsaturated precolouring of G with candidate set W that colours $W \cup N(W)$. After this, as $d(v) \leq m(G) - 1$, for all $v \in V \setminus (W \cup N(W))$, we know that this partial colouring can be extended to the entire graph (there exists a colour that does not appear in $N(v)$).

Let G' denote the induced subgraph $G[W \cup N[W]]$ and let H be a connected component of G'. A subset $R \subseteq V(H)$ is a *tight set of* H if $H[R]$ is connected and $N(u) \setminus W \subseteq R$, for every $u \in W \cap R$. R is called a *basic tight set of* H if either $R = V(C) \cup \bigcup_{u \in V(C) \cap W} (N(u) \setminus W)$, for some cycle C of H, or $R = (N(u) \setminus W) \cup \{u\}$, for some $u \in W \cap V(H)$. We denote the basic tight set $V(C) \cup \bigcup_{u \in V(C) \cap W} (N(u) \setminus W)$ by $[C]$ and $(N(u) \setminus W) \cup \{u\}$ by $[u]$.

The general idea is to colour each vertex of W with a different colour and, then, colour each connected component H of G' separately, using a sequence of tight sets of H, $R_1, \cdots, R_k$, where R_1 is basic, $R_i \subset R_{i+1}$, $i = 1, \cdots, k-1$, and $R_k = V(H)$. So, we start by colouring R_1 and, at step i, we extend the precolouring that colours R_{i-1} to a precolouring that colours R_i, $i = 2, \cdots, k$.

In the next subsection, we show how to obtain this sequence. We also ensure some other properties for the tight sets of the sequence that will be important for colouring G'. Then, in Subsection 6.2 we show how to colour a basic tight set of H and how to extend the precolouring that colours R_{i-1} to a precolouring that colours R_i. We will also need the following definitions.

Let H be a connected component of G'. Given a tight set R of H, we say that X is an *R-flap* if X is the set of vertices of a connected component of $H \setminus R$. Also, if $u \in N^W(R)$ is such that $N(u) \setminus (W \cup R) \neq \emptyset$, we say that u is an *intermediate vertex of R*.

6.1. Tight sets

Before we explain how to obtain the desired sequence of tight sets, we show how to obtain a basic tight set having a convenient property. To better understand the necessity of this property, consider a connected component H of G' and a sequence of tight sets $R_1, \cdots, R_k$ of H as mentioned before. Suppose that we have an unsaturated precolouring ψ with candidate set W that colours exactly $R_i \cup W$ in H, $i \in \{1, \cdots, k-1\}$. In order to extend ψ to colour $R_{i+1} \setminus (R_i \cup W)$, we would like to ensure that there are sufficiently many vertices of W "distant" from any R_i-flap X, so that we can use the colours of these "distant" vertices to colour $(X \cap (R_{i+1} \setminus R_i)) \setminus W$. More formally, we want to ensure the following for all tight set R in the sequence:

(Half Property) $|X \cap W| \leq \frac{1}{2}|W|$, for every R-flap X.

From now on, we write "R satisfies (HP)" when a tight set R satisfies the Half Property. Observe that if R is a tight set of H that satisfies (HP) and R' is a tight set of H containing R, then R' also satisfies (HP). Thus, we need only to ensure that the first tight set of the sequence satisfies (HP). We prove the existence of a basic tight set that satisfies (HP) in the following lemma, where we also ensure another property that will be useful later.

Lemma 19. *Let H be a connected component of G'. There exists a basic tight set R that satisfies (HP). Furthermore, if $R = [C]$, for some cycle C in H, then there is no $u \in W$ such that $R \subseteq (N(u) \setminus W) \cup \{u\}$.*

Proof: We first prove that there exists a basic tight set of H that satisfies (HP). Observe that, as $V(G') = N(W) \cup W$, H must have at least one basic tight set. So, let R be a basic tight set of H and $\{X_1, \ldots, X_k\}$ be the set of R-flaps with indices such that $|X_1 \cap W| \geq \cdots \geq |X_k \cap W|$. If $|X_1 \cap W| \leq \frac{1}{2}m(G)$, we are done; so, suppose otherwise. Observe that $|(V(H) \setminus X_1) \cap W| \leq \frac{1}{2}m(G)$. In the procedure described in the next two paragraphs, we obtain a basic tight set R' of H such that, for every R'-flap

X, either (I) $X \subseteq V(H) \setminus X_1$, in which case $|X \cap W| \leq \frac{1}{2}m(G)$; or (II) $X \subset X_1$. Thus, if R' still has a flap X containing more than $\frac{1}{2}m(G)$ vertices of W, then $|X| < |X_1|$. So, as the graph is finite, we can run the procedure until we find the desired tight set.

Let $N = |N(X_1) \cap R|$. We know that $|N| \leq 2$ and N separates R from X_1. Suppose that there exists a cycle C intersecting R and X_1 and let $R' = [C]$. Trivially, R' is a basic tight set of H and, as G is a cactus, C contains N. Consequently, either (I) or (II) holds, for every R'-flap. Now, suppose that there is no such cycle. Then, we know that $N(X_1) \cap R = \{x\}$ and $N(x) \cap X_1 = \{x'\}$. Also, x' separates X_1 from R and, trivially, if $x' \in W$, then every $[x']$-flap satisfies (I) or (II). So, suppose that $x' \notin W$. In this case, $x \notin W$ as, otherwise, x' should be in R. Also, as $V(H) \subseteq N(W) \cup W$, there must exist $v \in N^W(x')$, and every $[v]$-flap satisfies (I) or (II).

Now, let R be a basic tight set of H satisfying (HP). If $R = [C]$, for some cycle C in H, and $R \subseteq (N(u) \setminus W) \cup \{u\}$, for some $u \in W$, then $R \subseteq [u]$ and, consequently, $[u]$ is a basic tight set that satisfies the lemma. $\square$

Now, we want to construct a desired sequence from a basic tight set R satisfying the lemma above. So, we set R_1 to R and, while the current set R_i is not equal to $V(H)$, we obtain R_{i+1} from R_i by adding either $(N(u) \setminus W) \cup \{u\}$, for some intermediate vertex u of R_i, or $(N(R_i) \cap X) \setminus W$, for some R_i-flap X, in the case R_i has no intermediate vertex. The following two lemmas prove that this procedure works.

Lemma 20. *Let H be a connected component of G' and R be a tight set of H that satisfies (HP). Then $w \in R$, for all $w \in W$ such that $N(w) \setminus W \subseteq R$.*

Proof: Let $w \in W \setminus R$. Observe that if w is not in the same connected component as R and $N(w) \setminus W \subseteq R$, then $N(w) \setminus W = \emptyset$ and, as $d(w) = m(G) - 1$, we have $W = N[w]$, a contradiction since $V(H) \cap W \neq \emptyset$ (recall that $V(H) \subseteq V(G') \subseteq W \cup N(W)$). So, let X be the R-flap containing w and denote by S the set $N(w) \cap R$. By Lemma 2, we know that $|S| \leq 2$, and, by (HP) and the fact that $m(G) \geq 7$, there must exist at least 4 vertices in $W \setminus X$. Observe that, as $d(w) \geq m(G) - 1$, for each vertex in $W \setminus \{w\}$

non-adjacent to w, there must exist at least one vertex in $N(w) \setminus W$, i.e.,

$$\begin{aligned}
|N(w) \setminus W| &\geq |W \setminus N[w]| \\
&\geq |W \setminus (S \cup X)| \\
&= |(W \setminus X) \setminus S| \\
&= |W \setminus X| - |S \cap W|
\end{aligned}$$

Also, as $|W \setminus X| \geq 4$ and $N(w) \setminus W = (N(w) \setminus (S \cup W)) \cup (S \setminus W)$, we have:

$$|N(w) \setminus (W \cup S)| \geq 4 - (|S \cap W| + |S \setminus W|) = 4 - |S| \geq 2$$

Thus, $N(w) \setminus (W \cup R) \neq \emptyset$. $\square$

By the definition of tight set, we know that $R \cup [u]$ is a tight set, for any intermediate vertex u of R; also, $R \subset R \cup [u]$ trivially holds. The following lemma proves that if R has no intermediate vertex, we can still grow the current tight set.

Lemma 21. *Let H be a connected component of G' and R be a non-empty tight set of H, $R \neq V(H)$. If R has no intermediate vertex, then $R' = R \cup N^X(R)$ is a tight set, for any R-flap X. Furthermore, $R \subset R'$.*

Proof: Let X be any R-flap, $S = N^X(R)$ and $R' = R \cup S$. Obviously, $R \subset R'$ and R' is still connected. Additionally, by Lemma 20 and the fact that R has no intermediate vertex, we know that $S \cap W = \emptyset$ and, consequently, $N^S(u) = \emptyset$, for all $u \in W \cap R$. Thus, $W \cap R = W \cap R'$ and $N^{R'}(u) = N^R(u)$, for all $u \in R \cap W$. So, R' is also tight. $\square$

6.2. Colouring Phase

Let G be a cactus with $m(G) \geq 7$, $W \subseteq D(G)$ be a quasi-good set of G and $G' = G[W \cup N(W)]$. We say that a precolouring ψ of G is *nice for W* (or simply nice, if there is no ambiguity) if it is an unsaturated precolouring with candidate set W that colours only vertices of G' and is such that, for every connected component H of G', the coloured vertices in H are exactly $(W \cap V(H)) \cup R$, where R is either empty or is a tight set of H that satisfies (HP). For simplicity, as W is the candidate set of ψ and must be coloured, we say only that ψ colours R. Also, from now on, we consider that the vertices of W are coloured with colours from $\{1, \cdots, m(G)\}$ and we denote the vertex of W coloured with i by w_i.

So, given a nice precolouring ψ of G' and a connected component H such that ψ colours only vertices of W in H, we will pick a basic tight set R as explained in Lemma 19, extend ψ to colour R, obtaining a nice precolouring ψ^+, then we extend ψ^+ to colour: $N(u) \setminus W$, for some intermediate vertex u of R; or, if R has no intermediate vertex and $R \neq V(H)$, we extend ψ^+ to colour $N^X(R)$, for some R-flap X. As these steps sometimes intersect, for a better understanding of the general outline of the proof, we first state the lemmas and postpone their proofs to later in the text. In the following, consider G to be a cactus with $m(G) \geq 7$, $W \subseteq D(G)$ to be a quasi-good set of G and H to be a connected component of $G' = G[W \cup N(W)]$.

Lemma 22. *Let ψ be a nice precolouring that colours only vertices of H that are in W and $R = [w]$, for some $w \in H \cap W$. Then, there exists a nice precolouring that extends ψ and colours R.*

The colouring of a basic tight set $[C]$ is divided in two parts: first we colour $V(C)$, which is done in Lemma 23; then, we colour $N(w) \setminus (V(C) \cup W)$, for all $w \in V(C) \cap W$, which is done either in Lemma 23 or in Lemma 24.

Lemma 23. *Let $R \subseteq H$ be basic tight set satisfying Lemma 19 such that $R = [C]$, for some cycle C of H, and ψ be a nice precolouring that colours only vertices of W in H. Then, we can extend ψ to an unsaturated precolouring that colours either R, if $|R \cap W| = 1$, or $V(C)$, otherwise.*

Lemma 24. *Let R be a basic tight set satisfying Lemma 19 such that $R = [C]$, for some cycle C of H, and ψ be an unsaturated precolouring that colours $V(C)$. Then we can extend ψ to a nice precolouring that colours R.*

Before we present the next lemma, we need some more definitions. Let R be either the vertex set of a cycle in H or a tight set of H and let ψ be an unsaturated precolouring that colours R. Let $x \in V(H) \setminus R$; we say that colour i is *forbidden for x in ψ* if there exists $w \in N^W(x) \cup \{x\}$ such that w has a neighbour coloured with i; we denote the set of colours forbidden for x in ψ by $F_\psi(x)$ (we ommit ψ, if there is no ambiguity). If R is the vertex set of a cycle in H, let $u \in R \cap W$ and X be the vertex set of all connected components of $H - u$ containing some $x \in N(u) \setminus (R \cup W)$; otherwise, let u be an intermediate vertex of R and X be the R-flap containing u. Finally, let U be the set of uncoloured neighbours of u (trivially, $U = N(u) \setminus (W \cup R)$) and Q be the bipartite graph $(U \cup M(u), E')$, where $(x, c) \in E'$ if and only if $c \notin F(x)$. The following lemma proves the existence of a matching in Q that covers $M(u)$ which is used to extend ψ.

Lemma 25. *Let R, ψ, u, X and Q defined as above. If $N^Q(c) \neq \emptyset$, for all $c \in M(u)$, and there exist $w_{c_1}, w_{c_2} \in W \setminus X$ such that $c_1, c_2 \in M_\psi(u)$ and $c_1, c_2 \notin F_\psi(x)$, for all $x \in N(u) \setminus (W \cup R)$, then we can extend ψ to colour $R \cup [u]$.*

Finally, the following lemma extends a precolouring that colours R in the case where R is a tight set with no intermediate vertex.

Lemma 26. *Let R be a tight set of H and ψ be a nice precolouring that colours R. If R has no intermediate vertex and $R \neq V(H)$, then there is a nice precolouring of G that extends ψ and colours $R' = R \cup N^X(R)$, for some R-flap X.*

Proof of Theorem 18: First, colour each $u \in W$ with a different colour from $\{1, \cdots, m(G)\}$, obtaining ψ. Then, for each connected component H of $G' = G[W \cup N(W)]$, we extend ψ to colour $V(H)$ as follows. Let R be a basic tight set obtained in Lemma 19. If $R = [u]$, for some $u \in W$, then use Lemma 22 to extend ψ to colour R; otherwise, use Lemma 23 (and Lemma 24, if necessary) to colour R. Let ψ^+ be the obtained precolouring.

Now, if R has no intermediate vertex, we can apply Lemma 26 to extend ψ^+ to colour $R \cup N^X(R)$, for any R-flap X, which is a tight set by Lemma 21. So, suppose that R has an intermediate vertex u and let X be the R-flap containing u. Also, let U and Q be defined as in Lemma 25. We want to prove that we can use Lemma 25 to extend ψ^+:

1. There exist $w_{c_1}, w_{c_2} \in W \setminus X$ such that $c_1, c_2 \in M(u)$ and $c_1, c_2 \notin F(x)$, for all $x \in N^X(u)$: since R satisfies (HP), we have $|W \setminus X| \geq 4$; also, we know that $|N(u) \cap R| \leq 2$. So, let $L = \{w_{d_1}, w_{d_2}, w_{d_3}, w_{d_4}\} \subseteq W \setminus X$. If $|N(u) \cap R| = 2$, then, as u separates X from R, at least two colours $d_i, d_j \in L$ are in $M(u)$ and are not in $F(x)$, for all $x \in N^X(u)$. Otherwise, at most one colour $d_i \in L$ is such that $d_i \notin M(u)$. Also, at most one $x \in N^X(u)$ may have a neighbour in R and it has at most one such neighbour; hence, at most one other colour $d_j \in L \setminus \{d_i\}$ appears in $N(x)$, i.e., $L \setminus \{d_i, d_j\} \subseteq M(u)$ and $(L \setminus \{d_i, d_j\}) \cap F_{\psi^+}(x') = \emptyset$, for all $x' \in N^X(u)$;

2. $N^Q(c) \neq \emptyset$, for all $c \in M(u)$: consider $c_1, c_2 \in M(u)$ satisfying the previous item and let $c \in M(u) \setminus \{c_1, c_2\}$. As ψ^+ is unsaturated, we know that $|U| \geq 3$. Note that c has at most three non-neighbours in U, in which case: (I) there exist $x, y, z \in U$ such that x, y, w_c are in the same component of $H - u$ and there exists $w \in N^W(z) \cup \{z\}$ such that w has a neighbour in $R \setminus W$ coloured with c. But then, as $|W \setminus X| \geq 4$, there must exist $w_{c'} \in W \setminus \{w_{c_1}, w_{c_2}\}$ such that $c' \in M(u)$. As $w_c \in X$ (i.e., $c \neq c'$), we have $|M(u)| \geq 4$ and, consequently, c must have at least one neighbour in Q.

We can then apply Lemma 25 to extend ψ^+ to colour $R \cup [u]$, which is trivially a tight set of H. We repeat this procedure until having coloured all the vertices of G'. After this, as W is a good set, $d(x) < m(G) - 1$, for all $x \in V(G) \setminus V(G')$; hence, there exists a colour $c \in \{1, \cdots, m(G)\}$ that does not appear in $N(x)$ and the precolouring can be extended to a b-colouring of G with $m(G)$ colours. □

We now present the proofs of the lemmas.

Proof of Lemma 22: Let $X_1, \cdots, X_q$ be the vertex sets of the non-trivial connected components of $H - w$ containing at least one vertex of $[w]$ (i.e., $|X_i| \geq 2$ and $X_i \cap (N(w) \setminus W) \neq \emptyset$). Observe that if $x \in N(w) \setminus (W \cup \bigcup_{i=1,\cdots,q} X_i)$, then $\{x\}$ is a connected component of $H - w$. So, after colouring $N(w) \cap X_i$, for all $i \in \{1, \cdots, q\}$, we can give any colour from $M(w)$ to x, if there exists such a colour; otherwise, i.e., if $M(w) = \emptyset$, we can colour x with any colour different from $\psi(w)$. An analogous argument can be made in the case where $X_i \cap W = \{u\}$ and $N(w) \cap X_i = \{u, x\}$, $x \neq u$: since w is the only coloured neighbour of u, we have $M(w) \subseteq M(u)$; thus, after colouring $X_j \cap N(w)$, for all $j \in \{1, \cdots, q\}$, $j \neq i$, we can give any colour from $M(w)$ to x (again, if $M(w) = \emptyset$, colour x with any colour different from $\psi(w), \psi(u)$ - exists as $m(G) \geq 7$). So, suppose that $(X_i \cap W) \setminus N(w) \neq \emptyset$, for all $i \in \{1, \cdots, q\}$. By Lemma 2, we know that $|X_i \cap [w]| \leq 2$, for all $i \in \{1, \cdots, q\}$. So, consider, without loss of generality, that there exists an index $p \in \{0, \cdots, q\}$ such that $|X_i \cap [w]| = 2$, for all $i \in \{1, \cdots, p\}$, and $|X_i \cap [w]| = 1$, for all $i \in \{p+1, \cdots, q\}$. For each $i \in \{1, \cdots, q\}$, denote the vertices in $X_i \cap [w]$ by x_i, y_i (if $i > p$, consider $x_i = y_i$). Also, denote the set $\{x_1, y_1, \cdots, x_q, y_q\}$ by Z. We want to construct a function $f : Z \to M(w)$ in such a way that the vertex of W coloured with $f(z_i)$ is in X_i, for all $z_i \in Z$. Then, we will use this function to colour $[w]$. So, consider the cases:

- $i > p$: let $u \in (W \cap X_i) \setminus N(w)$ and set $f(x_i)$ to $\psi(u)$;
- $i \leq p$: if there exist $u_1, u_2 \in W \cap X_i$ such that u_1 is reachable from x_i and not from y_i and u_2 is reachable from y_i and not from x_i, then set $f(x_i)$ to $\psi(u_1)$ and $f(y_i)$ to $\psi(u_2)$. Otherwise, suppose, without loss of generality, that every vertex of $W \cap X_i$ reachable from y_i is also reachable from x_i. Let $u \in W \cap X_i$ reachable from both x_i and y_i, if there exists one, or let u be any vertex in $W \cap X_i$, otherwise. Set $f(x_i)$ to $\psi(u)$ and $f(y_i)$ to *null*.

Now, let $J = \{z \in Z : f(z) \neq null\}$. Note that $x_i \in J$, for all $i \in \{1, \cdots, q\}$, and that $f(z) \neq f(z')$, for all $z, z' \in J$. Furthermore, let $z, z' \in J$ and $f(z) = c$; we know that w_c is not adjacent to w and is not reachable from z'. Thus, if $|J| \geq 2$, we can permute the colours defined by f on the vertices of J in such a way that $\psi(z) \neq f(z)$, for all $z \in J$, and obtain an unsaturated extension of ψ that colours J. So, suppose that $|J| = 1$ (hence, $q = 1$). If $p = 0$, as x_1 is not encircled by W, there must exist $u \in W$ not reachable from x_1 and we can color x_1 with $\psi(u)$. So, consider $p = 1$. If there exists $u \in X_1 \cap W$ not reachable from y_i (recall the construction of f), then colour y_i with $\psi(u)$ and x_i with $\psi(u')$, for any $u' \in W$ not reachable from x_i (exists, as x_i is not encircled by W). So, suppose that every $u \in X_i \cap W$ is reachable from both x_i and y_i. We know that there exists a cycle C containing x_1, y_1, w and at least one $u \in W \cap X_i$. Trivially, any $V(C)$-flap X separated from C by w is also a connected component of $H - w$ and, as $q = 1$, we know that $|W \cap X| \leq 1$. Furthermore, let A be the subset of $[w]$-flaps containing any neighbour of x_1 or y_1. Note that any $V(C)$-flap separated from C by other vertex than w is contained in some $[w]$-flap in A. One can then verify that if $[w]$ satisfies (HP) or has a 4-gap, then $V(C)$ also does; thus, as $V(C) \nsubseteq [v]$, for all $v \in W$, we can consider the basic tight set $V(C)$ instead of $[w]$.

Denote by ψ^+ the extension of ψ obtained in the previous paragraph. Now, let $S = Z \setminus J$ (subset of uncoloured y_i's) and consider $y_i \in S$ (note that if $q = 1$, then $S = \emptyset$). Suppose that there exists $u \in X_i \cap W$ not reachable from y_i. If $f(x_i) \neq \psi^+(u)$, then colour y_i with $\psi^+(u)$ and remove it from S. Otherwise, by the construction of f, we have that $W \cap X_i = \{u\}$. Thus, $\psi^+(N(y_i)) \subseteq \psi^+(\{x_i, w\})$ and we can colour y_i either with a colour from $M_{\psi^+}(w)$, if one exists, or with any colour not in $\psi^+(N(y_i))$. So, we denote the subset $X_i \cap W$ by F_i and consider that every vertex in F_i is reachable from both x_i and y_i, for all $y_i \in S$. Also, note that $N(y_i) \cap [w] = \{w\}$, for all

$y_i \in S$; thus, during the colouring of $[w]$, every coloured neighbour of y_i is in W, for all $y_i \in S$. Consequently, if $|S| > |M_{\psi^+}(w)|$ and we are able to colour $S' \subset S$ with cardinality $|M_{\psi^+}(w)|$ each with a different colour from $|M_{\psi^+}|$, then we can colour y_i with any colour not in $N(y_i)$, for all $y_i \in S \setminus S'$ (such a colour exists as $q \geq 2$). So, from now on we consider that $|M(w)| \geq |S|$. Trivially, $F_i \cap F_j = \emptyset$, for every pair $y_i, y_j \in S$. Let $u \in W$ be such that $\psi^+(u) \in M(w)$. We know that if some y_i cannot be coloured with $\psi^+(u)$, then $u \in F_i$ and, consequently, y_j can be coloured with $\psi^+(u)$, for every $y_j \in S \setminus \{y_i\}$. So, if $|S| \geq 2$ and $M(w) \setminus \psi^+(F_i) \neq \emptyset$, for every $y_i \in S$, then we can colour S with colours from $M(w)$. Now, suppose otherwise and consider, without loss of generality, that $y_1 \in S$ and $M(w) \subseteq \psi^+(F_1)$. As y_1 is not encircled by W, there must exist a vertex $u \in W$ not reachable from y_1 (and, consequently, not in X_1). As $u \notin F_1$ and $M(w) \subseteq \psi^+(F_1)$, we must have that $\psi^+(u) \notin M(w)$. So, let $z \in N(w)$ be such that $\psi^+(z) = \psi^+(u)$. If $z = u$, then $d(w) \geq m(G)$ (as u is not reachable from y_1) and we can repeat the colour $\psi^+(z)$ in y_1; and if $z \neq x_1$, then colour z with c, for any $c \in M(w)$, and y_1 with $\psi^+(u)$. So, suppose that $z = x_1$. Recall the definition of W_x and note that $W_x = W_y$. Thus, as W does not encircle the pair (x_1, y_1), either $d(w) \geq m(G)$, in which case we can colour y_1 with $\psi^+(u)$, or there exists $u' \in W \setminus (W_x \cup \{u\})$. In this case, as $\psi^+(x_1) \neq \psi^+(u')$, we can apply the same argument as before to colour y_1 with $\psi^+(u')$. After this, just colour the remaining uncoloured vertices in $[w]$ with the colours missing in $N(w)$. $\square$

Proof of Lemma 23: First, note that $R \cap W \subseteq V(C)$ and $N^R(x) \subseteq V(C)$, for all $x \in R \backslash W$. For each $x \in V(C) \backslash W$, denote by $N^*(x)$ the subset $N^W(x) \backslash R$. Note that if $N^W(x) \cap R = \emptyset$, then $N^*(x) \neq \emptyset$. If $R \cap W = \emptyset$, it is easy to use the colours in $\psi(\bigcup_{x \in R} N^*(x))$ to colour R. So, consider $R \cap W = \{w\}$ and let $C = (w, x_1, \ldots, x_q)$. We can colour $[w]$ using Lemma 22, obtaining ψ^+. Then, let $\psi^+(w) = c$, $\psi^+(x_1) = c_1$ and $\psi^+(x_q) = c_q$. Colour x_i with c, for each i even in $\{2, \cdots, q-1\}$. Then, for each i odd in $\{3, \cdots, q-1\}$, if w_{c_1} is not reachable from x_i, then colour x_i with c_1; otherwise, colour x_i with $\psi(w')$, for any $w' \in N^*(x_{i-1})$. At the end, if q is even and $\psi(x_{q-1}) = c_1 = c_q$, then change the colour of x_q to $\psi(w')$, for any $w' \in N^*(x_{q-1})$; otherwise, if q is even and $\psi(x_{q-1}) = c_q \neq c_1$ (in which case, we know that w_{c_1} is separated from R by x_{q-1} and $w_{c_q} \in N^*(x_{q-2})$), then change the colour of x_1 to c_q and of x_q to c_1. So, from now on, we suppose that $|W \cap R| \geq 2$.

First, consider that there exists at least one maximal subpath $P \subseteq H[V(C)]$ such that $P \cap W = \emptyset$ and P has length greater than one. So, let $P_1, \cdots, P_q$ be all such subpaths and let x_i, y_i be the extremities of P_i, for every $i \in \{1, \cdots, q\}$. We know that $N^*(x) \neq \emptyset$, for all $x \in P_i \setminus \{x_i, y_i\}$, $i \in \{1, \cdots, q\}$. We first colour $S = V(C) \setminus \bigcup_{i=1}^{q}(P_i \setminus \{x_i, y_i\})$. Let P be a connected component of $H[S]$. Trivially, P is a path; so, let z, z' be the extremities of P. We colour P as follows (we apply the first step until it is not possible anymore, then we apply the second step):

- Let $w \in P \cap W$ non adjacent to z or z' and $N^P(w) = \{t_1, t_2\}$. If $t_i \notin W$, let $w_i \in (W \cap P) \setminus \{w\}$ closest to t_i, $i = 1, 2$. We know w_i exists as $t_i \neq z, z'$; also, we know that t_i is within a link between w and w_i, $i = 1, 2$. If both t_1 and t_2 are not in W, then colour t_1 with $\psi(t_2)$, if t_1 is not coloured yet, and t_2 with $\psi(t_1)$, if t_2 is not coloured yet. Otherwise, suppose that $t_1 \in W$, $t_2 \notin W$ and t_2 is not coloured (if both are in W or t_2 is coloured, there is nothing to do). If (t_2, w_2) is not an edge, than colour t_2 with $\psi(w_2)$. Otherwise, let $t' \in N^P(w_2) \setminus \{t_2\}$. If $t' \in W$, as t_2 is not encircled by W, there must exist $w' \in W$ not reachable from t_2, in which case we colour t_2 with $\psi(w')$. Otherwise, we colour $t', t_2 \in N^P(w_2) \setminus W$ either in this step or in the next, in the case where $t' \in \{z, z'\}$;

- Let $w \in W \cap P$ adjacent to z or z' and $N^P(w) = \{t_1, t_2\}$, where $t_2 \in \{z, z'\}$. We know that t_2 is an extremity of some P_i, $i \in \{1, \cdots, q\}$. Let $x \in N(t_2) \cap P_i$ and w' be any vertex in $N^*(x)$. If $t_1 \in W$, then colour t_2 with $\psi(w')$; so, suppose otherwise. First, consider that $P \cap W = \{w\}$; then, t_1 is also the extremity of some P_j, $j \in \{1, \cdots, q\}$. Note that, as $|W \cap R| \geq 2$, we have $i \neq j$. Thus, colour t_2 with $\psi(w')$ and t_1 with $\psi(w'')$, for any $w'' \in N^*(x')$, where $x' \in N(t_1) \cap P_j$. Now, suppose $|P \cap W| \geq 2$ and define w_1 related to t_1 as in the previous step. If $w_1 \notin N(t_1)$, then colour t_1 with $\psi(w_1)$ and t_2 with $\psi(w')$. Otherwise: if t_1 is already coloured, colour t_2 with $\psi(w')$; otherwise, colour t_1 with $\psi(w')$ and t_2 with $\psi(w_1)$. Observe that this last case is the sole case where we colour t_2 with a colour in $\psi(W \cap R)$ (I).

Now, we colour $P_i \setminus \{x_i, y_i\}$, for all $i \in \{1, \cdots, q\}$. Let ψ^+ be the precolouring obtained above and consider some $P_i = \langle x_i = v_1, v_2, \cdots, v_p = y_i \rangle$, $i \in \{1, \cdots, q\}$. Also, let $u_1 \in N^R(v_1) \setminus P_i$ and $u_2 \in N^R(v_p) \setminus P_i$; as $|W \cap R| \geq 2$, we know that $u_1 \neq u_2$. If $\psi^+(v_1) \neq \psi^+(u_2)$ and $\psi^+(v_p) \neq \psi^+(u_1)$, then we can easily colour $v_2, \cdots, v_{q-1}$ by alternating the colours $\psi^+(u_1), \psi^+(u_2)$ in P_i;

so, suppose otherwise. By (I), $R \cap W = \{u_1, u_2\}$ and $N(u_1) \cap N(u_2) \neq \emptyset$, i.e., $C = (u_2, x, u_1, v_1, \cdots, v_p)$. So, let $w \in N^*(v_2)$. If there exists $w' \in W \setminus \{w\}$ not reachable from x, then colour x with $\psi^+(w')$, v_1 and v_p with $\psi^+(w)$ and, then, we can again alternate the colours $\psi^+(u_1), \psi^+(u_2)$ in $P_i \setminus \{v_1, v_p\}$. Otherwise, we have $W \setminus \{w\} \subseteq W_x$ (hence, $p = 3$) and, as $m(G) \geq 7$, there must exist $w' \in W$ separated from v_2 by $\{x, u_1, u_2\}$ (and, obviously, $w' \neq u_1, u_2, w$). Thus, we colour x with $\psi^+(w)$, v_1 with $\psi^+(u_2)$, v_3 with $\psi^+(u_1)$ and v_2 with $\psi^+(w')$.

Now, suppose that every maximal subpath of $H[V(C)]$ that does not intersect W has length at most one. Let $R \cap W = \{u_1, \cdots, u_q\}$. We write C as $(u_1, x_1, y_1, \cdots, u_q, x_q, y_q)$ and assume that $x_i = u_i$ when $(u_i, u_{i+1}) \in E(G)$ and that $y_i = x_i$ when the path between u_i and u_{i+1} has length at most two. We analyse the following cases (recall that $q \geq 2$):

- $q \geq 5$: for $i = 1, \cdots, q$, if $x_i \neq u_i$ then give colour $\psi(u_{(i+3) \mod q})$ to x_i. After this, for each uncoloured y_i, let $j = (i+1) \mod q$. Then, choose any colour in $\psi(W \cap R) \setminus \{\psi(x_i), \psi(u_j), \psi(x)\}$, where x is the neighbour of u_j in C different from y_i, i.e., x is either x_j or $u_{(j+1) \mod q}$. See Figure 16 for a better understanding. Note that, as $x_1, \cdots, x_q$ are coloured first, if $x_i = y_i$, for some i, the colouring is still proper.

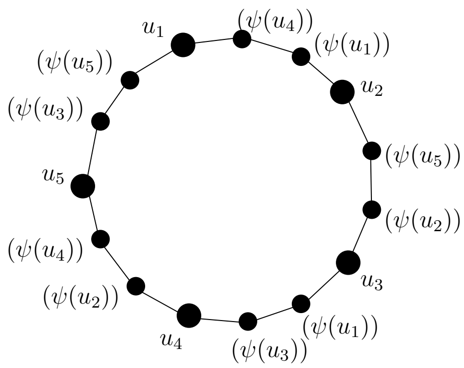

Figure 16: Representantion of a nice precolouring that colours $V(C)$ with $|V(C) \cap W| = 5$. Some of the "non-dense" vertices may not exist, i.e., the paths between vertices of W in the cycle may have length 1,2 or 3.

- $q = 4$: all the possible cycles are represented in Figure 17, as well as a precolouring of $R' \subseteq V(C)$. The only situations where there is some uncoloured vertex in R are in (a), (b) or (c). If (a) or (c) occurs, as x_1 is not encircled by W, there must exist a vertex $w \in W$ not reachable from x_3; then, just colour x_3 with $\psi(w)$. So, suppose that (b) occurs. If there exists $w \in (N^W(u_1) \cup N^W(x_1)) \setminus R$, then give colour $\psi(w)$ to x_3 and, as x_1 is not encircled by W, there must exist $w' \in W$ not reachable from x_1; then, give colour $\psi(w')$ to x_1. Otherwise, we can suppose that $(N(u_i) \cap W) \setminus R = \emptyset$, $i = 1, \cdots, 4$, and $(N(x_i) \cap W) \setminus R = \emptyset$, $i \in \{1, 3\}$. As $m(G) \geq 7$, we can choose any two colours $c, c' \in \{1, \cdots, m(G)\} \setminus \{\psi(u_1), \psi(u_2), \psi(u_3), \psi(u_4)\}$ to give to x_1 and x_3.

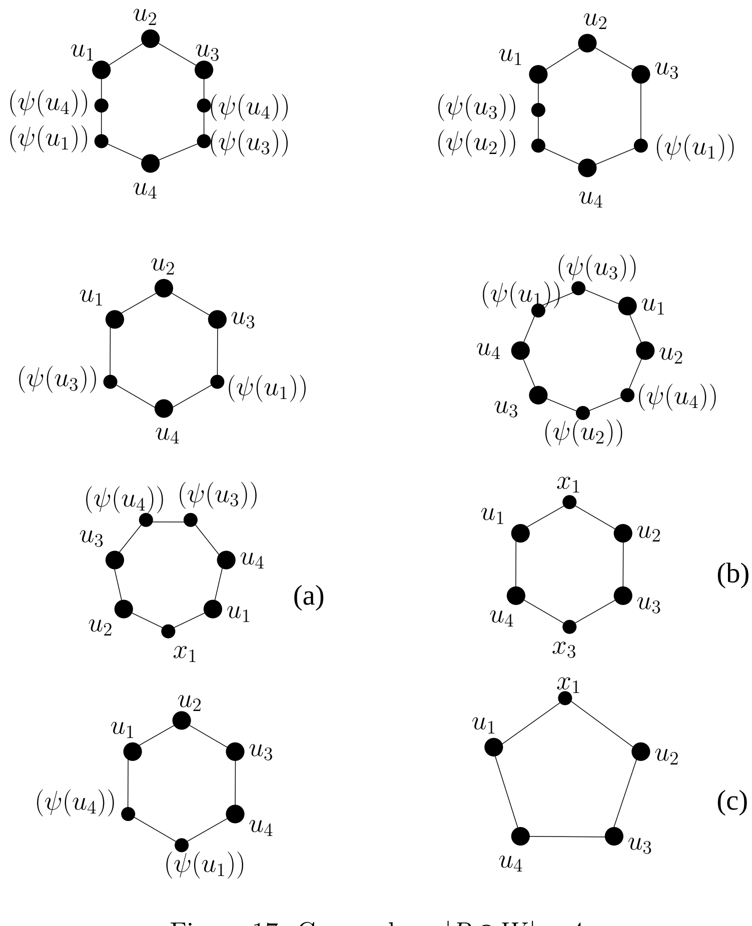

Figure 17: Cases where $|R \cap W| = 4$.

- $q = 3$: all the possible cycles are represented in Figure 18, as well as a precolouring of $R' \subseteq V(C)$. The only situations where there is some uncoloured vertex in C are in (a), (b), (c) and (d). If (d) occurs, as x_3 is not encircled by W, then there must exist $w \in W$ not reachable from x_3 and we can just give colour $\psi(w)$ to x_3. Now, suppose that (a) occurs. If there exists $w \in (N^W(x_2) \cup N^W(u_2)) \setminus V(C)$, then give colour $\psi(w)$ to y_2 and colour $\psi(u_1)$ to x_2. Otherwise, suppose that $(N^W(y_2) \cup N^W(u_3)) \setminus R$ is also empty (otherwise we have an analogous situation) and give colour $\psi(u_1)$ to y_2 and any colour from $M(u_2)$ to x_2 (such a colour must exist as $m(G) \geq 7$ and $\psi(N[u_2] \cup N(x_2)) = \{\psi(u_1), \psi(u_2), \psi(u_3)\}$). Now, suppose that (b) occurs. If there exists any $w \in W \setminus \{u_2\}$ not reachable from x_3, then give colour $\psi(w)$ to x_3. Otherwise, as x_3 is not encircled by W, we must have that u_2 is the only vertex in W not reachable from x_3; consequently, we have that $d(u_1) \geq m(G)$ and we can give colour $\psi(u_1)$ to y_2 and colour $\psi(u_2)$ to x_3. Finally, consider that (c) occurs. Observe that if we can colour x_3 with $\psi(w)$, for some w reachable from x_1 not through u_1, then, as x_1 is not encircled by W, there must exist w' not reachable from x_1 and we can colour x_1 with $\psi(w')$ (by the choice of w, we know that $w \neq w'$). So, we can suppose that $d(u_3) = m(G) - 1$ (otherwise, x_3 can be coloured with $\psi(u_2)$) and $(N^W(x_1) \cup N^W(u_2)) \setminus V(C) = \emptyset$. Analogously, we can suppose that $d(u_2) = m(G) - 1$ and $(N^W(x_3) \cup N^W(u_3)) \setminus V(C) = \emptyset$. Thus, if there exist $w, w' \in W \setminus (N[u_1] \cup \{u_2, u_3\})$, $w \neq w'$, then we can colour x_1 with $\psi(w)$ and x_3 with $\psi(w')$. Otherwise, as E2 does not occur, we have that either $W \setminus (N[u_1] \cup \{u_2, u_3\}) = \{w\}$ and $d(u_1) \geq m(G)$, in which case we colour x_1 and x_3 with $\psi(w)$, or $W \setminus \{u_2, u_3\} \subseteq N[u_1]$ and $d(u_1) \geq m(G) + 1$, in which case we colour x_1 and x_3 with any colour $c \notin \psi(\{u_1, u_2, u_3\})$.

- $q = 2$: recall that $N^C(u_1) = \{x_1, y_2\}$ and $N^C(u_2) = \{y_1, x_2\}$. First, suppose that at least one of the paths between u_1 and u_2 in C has length two, say $x_1 \neq y_1$, and that $u_1 \notin N(u_2)$. Then, give colour $\psi(u_2)$ to x_1 and $\psi(u_1)$ to y_1. Assume that $u_2 \neq x_2 \neq y_2$. If there exists $w \in N(x_2) \setminus V(C)$, then give colour $\psi(w)$ to y_2 and y_1 and colour $\psi(u_1)$ to x_2. Otherwise, if there exists $w \in N^W(u_2) \setminus V(C)$, then give colour $\psi(w)$ to y_2 and then colour x_2 with any colour in $M(u_2)$ (if there is none, just repeat colour $\psi(u_1)$ in x_2). Finally, suppose that $N^W(y) \setminus V(C) = \emptyset$, for all $y \in R$; then, we can pick two colours different from $\psi(u_1), \psi(u_2)$ to give to x_2 and y_2. Now, assume that $x_2 = y_2$.

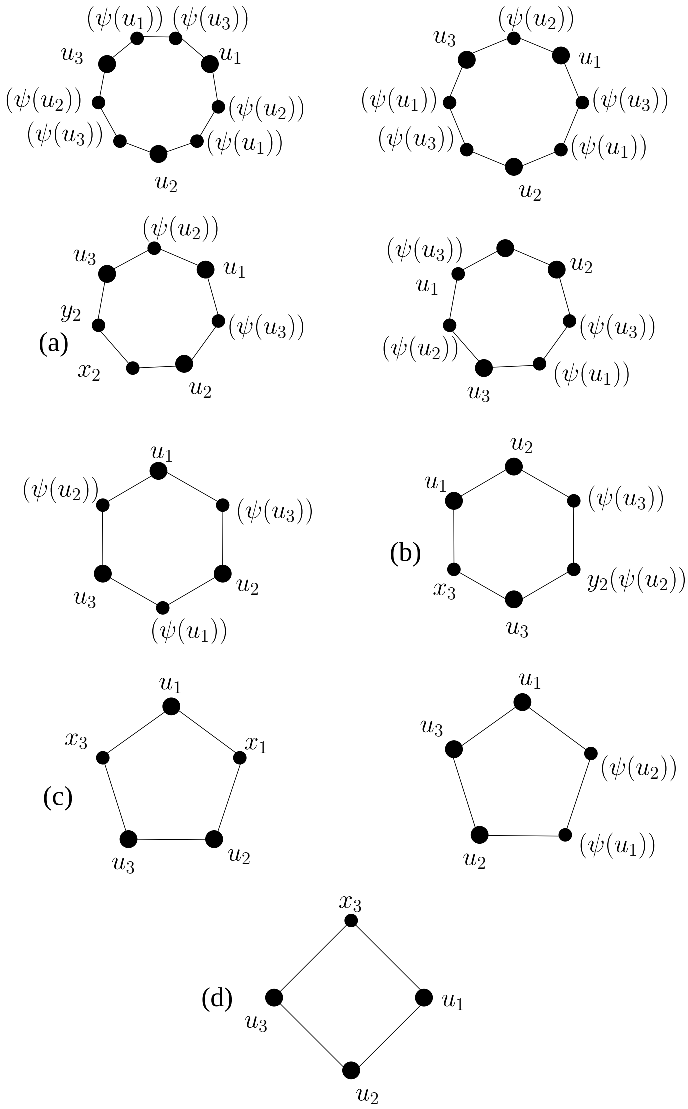

Figure 18: Cases where $|R \cap W| = 3$.

As x_2 is not encircled by W, there must exist $w \in W$ not reachable from x_2; so, give colour $\psi(w)$ to x_2. Now, consider $u_1 \neq x_1 = y_1$. If $x_2 \neq y_2$, we have an analogous situation; and if $(u_1, u_2) \in E(H)$, as x_1 is not encircled, there must exist some $w \in W$ not reachable from x_1 and we can colour x_1 with $\psi(w)$. So, it remains to consider the case where $u_1 \neq x_1 = y_1$ and $u_2 \neq x_2 = y_2$. We can again suppose that $N^W(x_i) \setminus V(C) = \emptyset$, $i = 1, 2$. If there exists $w, w' \in W \setminus N[u_1] \cup N[u_2]$, $w \neq w'$, then we can colour x_1 with $\psi(w)$ and x_2 with $\psi(w')$; so, suppose otherwise. If $W \setminus (N[u_1] \cup N[u_2]) = \{w\}$, then, as neither E1.a nor E1.b occurs, at least one of u_1, u_2, say u_1, is such that $N^W(u_1) \neq \emptyset$ and $d(u_1) \geq m(G)$. Thus, we can colour x_1 with $\psi(w)$ and x_2 with $\psi(w')$, for any $w' \in N^W(u_1)$. Now, suppose that $W \subseteq N[u_1] \cup N[u_2]$. As x_i is not encircled, for $i = 1, 2$, then at least one of u_1, u_2, say u_1, is such that $N^W(u_1) \neq \emptyset$ and $d(u_1) \geq m(G)$. If $d(u_1) = m(G)$, then, as E1.d does not occur, we have $N^W(u_2) \neq \emptyset$; consequently, as E1.c does not occur, then $d(u_2) \geq m(G)$ and we can colour x_1 and x_2 with one colour from $\psi(N^W(u_1))$ and one colour from $\psi(N^W(u_2))$. Now, assume that $d(u_1) > m(G)$. If $N^W(u_1)$ has more than one vertex, then we can colour x_1 and x_2 using colours from $\psi(N^W(u_1))$. Otherwise, as E1.b does not occur, we must have that $d(u_2) \geq m(G)$ and, again, we can use one colour from $\psi(N^W(u_1))$ and one colour from $\psi(N^W(u_2))$ to colour x_1 and x_2. $\square$

Proof of Lemma 24: Let $W \cap R = \{u_1, \cdots, u_q\}$ and, for each $i \in \{1, \cdots, q\}$, let J_i represent the set of vertices of all the connected components of $H - u_i$ containing some $x \in N(u_i) \setminus (V(C) \cup W)$. We claim that, if for some u_i there exist $w_{c_1}, w_{c_2} \in W \setminus (J_i \cup \{u_i\})$ such that $c_1, c_2 \in M(u_i)$, then we can extend ψ to colour $R \cup [u_i]$ by applying Lemma 25. Indeed, if this holds, we know that and $c_1, c_2 \notin F(x)$, for all $x \in N(u_i) \setminus (V(C) \cup W)$. Also, either $|M(u_i)| = 2$ (and a matching of Q that covers $M(u_i)$ obviously exists) or $|M(u_i)| \geq 3$ and, as $|U \setminus N^Q(c)| \leq 2$, for all $c \in M(u_i)$, we have $N^Q(c) \neq \emptyset$ (hence, the premisse of Lemma 25 holds). So, suppose that:

(I) $|M(u_i) \cap \psi(W \setminus (J_i \cup \{u_i\}))| \leq 1$, *for some* $i \in \{1, \cdots, q\}$.

Note also that if $N(u_i) \cap V(C) \subseteq W$, as the colours of $N(u_i) \setminus W$ have no influence over the colouring of $V(C)$, we can colour $[u_i]$ separately using Lemma 22. So, we also suppose that:

(II) $N^C(u_i) \setminus W \neq \emptyset$, *for all* $i \in \{1, \cdots, q\}$.

Note that $2 \leq q \leq 4$, by Lemma 23 and (I). We then analyse the cases:

- $q = 2$: by (I), we can suppose that $|M(u_1) \cap \psi(W \setminus (J_1 \cup \{u_1\}))| \leq 1$. Thus, as at most two colours in $\psi(W \setminus (J_1 \cup V(C)))$ appears in $N(u_1)$, we have (i) $|W \setminus (J_1 \cup V(C))| \leq 2$. So, as $m(G) \geq 7$, we have $|J_1 \cap W| \geq 3$; hence $|W \setminus (J_2 \cup \{u_2\})| \geq 4$ and we can apply Lemma 25 to colour $N(u_2)$ after colouring $V(C) \cup [u_1]$. So, consider ψ^+ obtained from ψ by uncolouring $V(C) \setminus W$ and applying Lemma 22 to colour $[u_1]$. After this, let $N^C(u_2) = \{x_1, x_2\}$. If $x_1, x_2 \notin N[u_1]$, as $|N^C(\{x_1, x_2\}) \setminus \{u_2\}| = 2$ and $|(J_1 \cup \{u_1\}) \cap W| \geq 4$, there must exist $w_{c_1}, w_{c_2} \in (J_1 \cup \{u_1\}) \cap W$ such that $c_1, c_2 \notin \psi(N(x_1) \cup N(x_2))$; then, we colour x_i with c_i, $i \in \{1, 2\}$. In the case where only one of x_1, x_2 is coloured ($|N[u_1] \setminus \{x_1, x_2\}| = 1$), we can make an analogous argument. Finally, if some $x \in V(C)$ is still uncoloured, as $m(G) \geq 7$ and by (i), there exists a colour $c \notin F(x)$ with which we can colour x.

- $q = 3$: note that if $|W \setminus (J_i \cup R)| \geq 2$, for all $i \in \{1, 2, 3\}$, then $|W \setminus (J_i \cup \{u_i\})| \geq 4$, contradicting (I). So, suppose, without loss of generality, $|W \setminus (J_1 \cup V(C))| \leq 1$. As $m(G) \geq 7$, we have $|J_1 \cap W| \geq 3$. Thus, for $i \in \{2, 3\}$, $|W \setminus (J_i \cup \{u_i\})| \geq 5$ and we can apply Lemma 25 to colour $N(u_i)$ after colouring $N(u_1)$. Let ψ^+ obtained from ψ by uncolouring $V(C) \setminus W$ and applying Lemma 22 to colour $[u_1]$. We colour the uncoloured vertices in C in a clockwise direction, starting from the vertex closest to u, x. Obviously, $x \notin N(u)$. Also, there are at most five colours with which x cannot be coloured, namely: the colours of the right neighbour w_r of x in C and the right neighbour of w_r in C, when $w_r \in W$; the colours of the left neighbour w_l of x in C and the left neighbour of w_l in C, when $w_l \in W$; and the colour of some eventual $w \in N^W(\{x, w_r, w_l\}) \setminus V(C)$ (recall that $|W \setminus (J_1 \cup V(C))| \leq 1$). Thus, as $m(G) \geq 7$, there exists a colour with which we can colour x.

- $q = 4$: note that if $W \setminus (\bigcup_{i=1}^{4}(J_i \cap W) \cup V(C)) \neq \emptyset$ or $J_i \cap W \neq \emptyset$ and $J_k \cap W \neq \emptyset$, for some $i, k \in \{1, \cdots, 4\}$, $i \neq k$, then $|W \setminus (J_l \cup \{u_l\})| \geq 4$, for all $l \in \{1, \cdots, q\}$, contradicting (I). So, suppose, without loss of generality, that $W \setminus V(C) \subseteq J_1$. In this case, $N^W(x) = \{u_i\}$, for all $x \in N(u_i) \setminus W$, $i \in \{2, 3, 4\}$; then, we can colour $N(u_i) \setminus W$ independently with colours from $M(u_i)$ after we colour $N(u_1)$, for every $i \in \{2, 3, 4\}$. So, consider ψ^+ obtained from ψ by uncolouring $V(C) \setminus W$ and applying

Lemma 22 to colour $[u_1]$. Now, write $C = (u_1, x_1, \cdots, x_r)$. Note that, as $W \setminus V(C) \subseteq J_1$, we have $N^H(x) \subseteq V(C)$, for all $x \in V(C) \setminus W$. Thus, as $\psi(N(u_i)) \subseteq \psi(W \cap V(C))$, for any u_i having a neighbour uncoloured in C, if we colour the uncoloured vertices in C starting from x_1 up to x_r, at any point we have that $F(x_i)$ has at most four colours, namely the colours in $\psi(\{x_{(i-1) \mod r}, x_{(i-2) \mod r}, x_{(i+1) \mod r}, x_{(i+2) \mod r}\})$. So, as $m(G) \geq 7$, there exists a colour with which we can colour x_i.

Proof of Lemma 25: Suppose, by contradiction, that there is no matching in Q that covers $M(u)$. By Hall's Theorem (we direct the reader to [3]), we know that there exists a subset $C \subseteq M(u)$ such that $|C| > |N^Q(C)|$. So, let C be such a subset. As $N^Q(c) \neq \emptyset$, for all $c \in M(u)$, we know that $|C| > 1$; also, as at most one colour $c \in M(u)$ has more than two non-neighbours in U, we have $|N^Q(C)| \geq |U| - 2$. However, as $U \subseteq N(c_i)$, $i = 1, 2$, we have that $c_1, c_2 \notin C$, i.e., $|C| \leq M(u) - 2$. But then, $|U| - 2 \leq |N^Q(C)| < |C| \leq |M(u)| - 2$, contradicting the definition of unsaturated precolouring.

Now, let $\mathcal{M}$ be a matching of Q that covers $M(u)$. Colour x with c, for each $(x, c) \in \mathcal{M}$. By the definition of $F(x)$, we know that this step does not repeat any colours in $N(w)$, for all $w \in W$. However, there may exist some $x \in U$ still uncoloured (this occurs when $|U| > |M(u)|$). If $N^W(x) = \{u\}$, just colour x with any colour not in $N(x)$ (such a colour exists as x has at most two coloured neighbours, namely u and some $y \in R \cup N(u)$). Otherwise, let $w \in N^W(x) \setminus \{u\}$. As $R \cup [u]$ is connected, by Lemma 2, we know that w has at most one other neighbour in $R \cup [u]$; also, if it w has a neighbour in $R \cup [u]$, then no other $w' \in N^W(x) \setminus \{u, w\}$ has neighbours in $R \cup [u]$. So, as at least one between c_1 and c_2, say c_1, does not appear in $N(w)$, we can colour x with c_1. $\square$

Proof of Lemma 26: Let X be any R-flap, $S_X = N^X(R)$ and $S_R = N^R(X)$. By Lemma 2, we know that $|S_X| \leq 2$ and $|S_R| \leq 2$. Thus, as ψ is nice, we have $|X \cap W| \leq \frac{1}{2}m(G)$ and, as $m(G) \geq 7$, there must exist at least two vertices w_1, w_2 in $W \setminus X$ such that $\psi(w_i) \notin \psi(S_R)$. So, we can give colours $\psi(w_1), \psi(w_2)$ to the uncoloured vertices in S_X, obtaining an unsaturated precolouring ψ^+. By Lemma 21, we know that $R' = R \cup N^X(R)$ is tight and, as R satisfies (HP), then R' also does; consequently, the ψ^+ is also nice. $\square$

7. Conclusion

We generalize the result on trees by Irving and Manlove for the cacti with m-degree at least 7. We also give an algorithm that finds an optimum b-colouring of such a cactus. In fact, we characterize the cacti that do not have a quasi-good set and show some graphs that, although having a quasi-good set, cannot be b-coloured with $m(G)$ colours (we call them anomalous). Then, we prove that if G does not have a quasi-good set or is anomalous, then $\chi_b(G) = m(G) - 1$. And finally we prove that if G has a quasi-good set and $m(G) \geq 7$ (thus G is not anomalous), then $\chi_b(G) = m(G)$. We conjecture that if G has a quasi-good set and G is not anomalous, then $\chi_b(G) = m(G)$. It remains to prove this for $m(G) \leq 6$. Observe that, if this is true, then $\chi_b(G) \geq m(G) - 1$, for all cactus G.

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APÊNDICE D

Results on defective cacti

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May 19, 2011

Notation:

- We say a vertex v is dense if $d(v) \geq m(G) - 1$.
- Let $D(G)$ be the set dense vertices of G.
- If ψ is a colouring of G and c is a colour in ψ, then let $S_\psi(c)$ denote the colour class c in ψ, i.e., the set of vertices of G with colour c in ψ.

1 Useful Results on Cacti

Lemma 1. *Let G be a cactus and U and U' be two disjoint subsets of $V(G)$. If $G[U]$ and $G[U']$ are connected, then U has at most two neighbours in U'.*

Proof. By contradiction, suppose that U has at least three neighbours in U'. Let Z' be a set of three neighbours of U in U' and T' be a minimal tree subgraph of $G[U']$ that contains Z'. Note that T' is either a path with both endpoints in Z' or the three vertices in Z' have degree one in T', there is a unique vertex with degree three and any remaining vertex has degree two. Let Z be a minimal subset of U containing at least one neighbour for each vertex in Z' and let T be a minimal tree subgraph of $G[U]$ that contains Z. Again, we note that T is either a path with both endpoints in Z, a single vertex or $|Z| = 3$ and the three vertices in Z have degree one in T, there is a unique vertex with degree three and the remaining vertices have degree two. Therefore, from T' and T and with the edges between Z' and Z we can build a subgraph of G that consists of two vertices with three paths between them. This subgraph contains two cycles that share an edge in G. $\square$

2 Dense Vertices in Defective Cacti

Let G be a cactus graph. We wish to prove that the b-chromatic number of G is always close to $m(G)$. To do so, we study the structure of a cactus G with $\chi_b(G) < m(G)$. We say that such a graph is defective. One problem arises when we wish to study minimal defective graphs. Suppose that G is a defective cactus with large $m(G)$, say $m(G) \geq 20$. It just happens that it is likely for G to include a defective graph H with $m(H)$ much smaller than $m(G)$. As an example of such a graph H, consider the graph in Figure 1. The problem is that H tells us nothing of why G is defective. To fix this problem, we define an m-defective graph. We say that G is an m-defective graph if $\chi_b(G) < m(G)$

and $m(G) = m$. We say that G is minimal m-defective if G is m-defective and every proper subgraph H of G is not m-defective, i.e., either $m(H) < m$ or $m(H) = m$ and $\chi_b(H) = m(H)$.

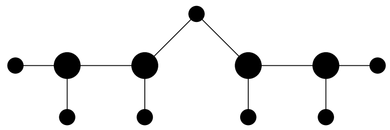


Figure 1: A defective graph.

Let G be a minimal m-defective cactus. The main results in this section are related to describing unnecessary vertices and edges in G. To be more precise, one result is to prove the following theorem.

Theorem 2. *If G is a minimal m-defective cactus and $m \geq 4$, then $|D(G)| = m$ and $d(u) = m - 1$ for $u \in D(G)$.*

From Theorem 2, we know that the dense vertices in G are incident to just enough edges for them to be dense and there are just enough vertices in $D(G)$ so that $m(G) = m$. With this idea in mind, one could expect that there were no edges between vertices not in $D(G)$. This is false due to the graphs in Figure 2 and in Figure 3. If H is one of the graphs in these figures, then consider $H' = H - uv$. Since H' is small, one can check that $\chi_b(H') = m(H')$. Actually, for any b-colouring of H' with $m(H')$ colours, u and v have the same colour. This implies that H is $m(H)$-defective. To see that H is minimal, note that for any $e \in E(H)$, either $e = uv$ and $\chi_b(H - e) = m(H)$ or $e \neq uv$ and $m(H - e) < m(H)$.

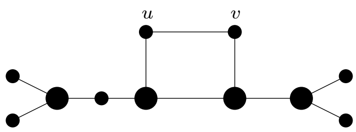


Figure 2: Anomalous graph with four dense vertices.

Say that a minimal m-defective cactus G is anomalous if there is an edge between two vertices not in $D(G)$. While we were incorrect to think there were no anomalous graphs due to the graphs shown, we were not off by much. To be more precise, our second main result in this section proves that these are precisely the anomalous minimal m-defective cacti.

Theorem 3. *Let G be a minimal m-defective cactus with $m \geq 4$. If G is anomalous, then G is isomorphic to a graph in Figure 2 or in Figure 3 and G has a unique edge between non-dense vertices.*

Throughout this section, we use many techniques based on recolouring a previously defined colouring. To simplify the presentation of these results, we define a recolouring function. Let ψ be a colouring of a graph G, A be a subset of $V(G)$ and c and c' be two colours in ψ. Define $\psi(A, c \leftrightarrow c')$ as the colouring

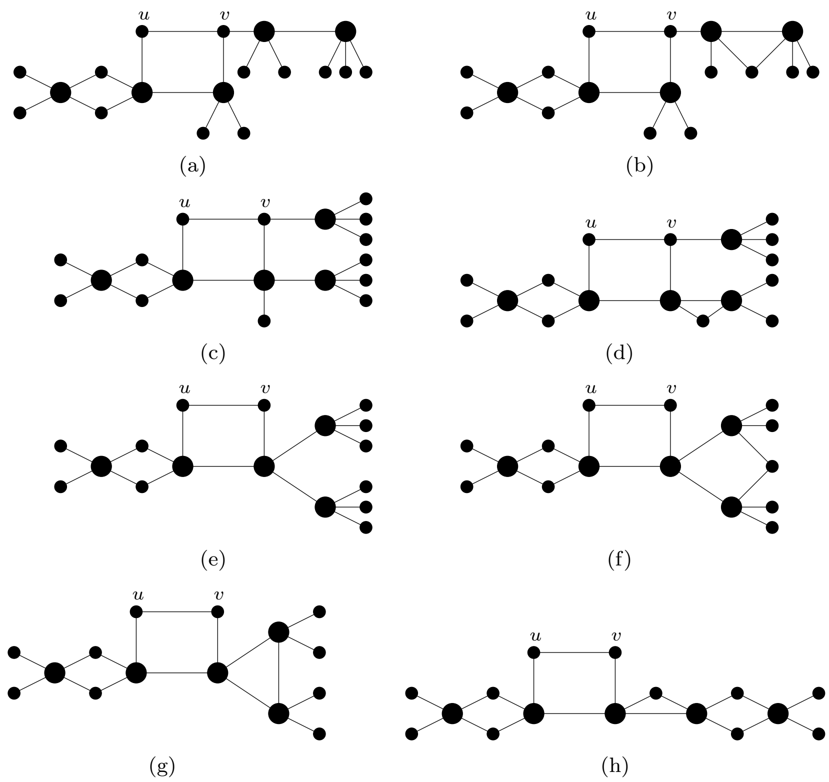


Figure 3: Anomalous graphs with five dense vertices.

obtained from ψ by exchanging the colours between the vertices in $S_\psi(c) \cap A$ and $S_\psi(c') \cap A$.

Lemma 4. *If G is minimal m-defective, then any vertex not in $D(G)$ is adjacent to at least one vertex in $D(G)$.*

Proof. By contradiction, suppose that $v \notin D(G)$ and v is not adjacent to a vertex in $D(G)$. Consider the graph G' obtained by deleting v from G. Since no vertex in $D(G)$ changed its degree in G', then $m(G') = m(G) = m$. This implies that $\chi_b(G') = m$ as G is minimal m-defective and G' is a proper subgraph of G. A b-colouring ψ of G' with m colours can be extended to G by colouring v with a colour not used in its neighbourhood. We can build this colouring as v has degree at most $m-1$ and, therefore, must miss at least one colour in ψ. $\square$

Lemma 5. *Let G be a minimal m-defective cactus, $w \in D(G)$ and C be a component of $G - w$. If C does not contain dense vertices, then $|V(C)| = 1$ and $d(w) = m - 1$.*

Proof. Suppose that C does not contain vertices in $D(G)$. Lemma 4 implies that any vertex in C is adjacent to w. Now, Lemma 1 implies that there are at most two vertices in C. If C contains an edge uv, let G' be obtained from G by deleting the edge uv. Since no vertex in $D(G)$ changed its degree in G', then

$m(G') = m(G) = m$. This implies that $\chi_b(G') = m$ as G is minimal m-defective and G' is a proper subgraph of G. Let ψ be a b-colouring of G' with m colours. If $\psi(u) \neq \psi(v)$, then ψ is also a b-colouring of G. Otherwise, we can recolour u to any other colour in ψ as w is adjacent to v with the same colour as u. In any case, we get a b-colouring of G with m colours. If $|V(C)| = 1$ and $d(w) \geq m$, then we can use a similar argument as before to show that a b-colouring of $G - C$ with m colours can be extended to a b-colouring of G by colouring the unique vertex in C with a colour different from w to get a contradiction. Thus, $d(w) = m - 1$. □

Lemma 6. *If G is a minimal m-defective cactus and $m \geq 4$, then $|D(G)| = m$.*

Proof. Let $u, v \in D(G)$ have maximum distance among pairs of dense vertices. Let C be a component of $G - u$ that contains at least one neighbour of u in G. Since $d(u) \geq 3$, there are at least two such components C by Lemma 1. Thus, consider that C does not contain v. We know that C does not contain any dense vertex as, otherwise, u and v would not have maximum distance among pairs of vertices in $D(G)$. Lemma 5 tells us that $d(u) = m - 1$ and u has a unique neighbour w in C. Let $H = G - w$. Note that $m(H) \leq m(G) = m$. By contradiction, suppose that $\chi_b(H) = m$ and there is a b-colouring ψ of H with m colours. We can build a b-colouring of G with m colours from ψ by giving a colour to w different from $\psi(u)$ to get a contradiction. Thus, $\chi_b(H) < m$ implies that $m(H) < m$, as G is minimal m-defective. Since u was the only vertex in $D(G)$ whose degree changed, we have that $|D(G - w)| = m - 1$ which implies the lemma. □

Lemma 7. *Let G be a minimal m-defective cactus with $m \geq 4$ and $w \in D(G)$. If $d(w) = m - 1$, then $N(w) \setminus D(G)$ is a stable set.*

Proof. By contradiction, suppose e is an edge betwen two neighbours of w not in $D(G)$. Note that $m(G - e) = m(G) = m$. Since G is minimal m-defective, then $\chi_b(G - e) = m$. Therefore, let ψ be a b-colouring of $G - e$ with m colours. Note that Lemma 6 implies that w is a b-vertex of ψ. Since $d(w) = m - 1$, all neighbours of w have distinct colours and ψ is also a b-colouring of G. □

Lemma 8. *Let G be a minimal m-defective cactus with $m \geq 4$, u and v be two vertices of $D(G)$ with $d(u) = d(v) = m - 1$ and C be a component of $G - v - u$. If C contains neighbours of both u and v and C contains no dense vertices, then $|V(C)| = 1$, i.e., C contains only a common neighbour between u and v.*

Proof. Suppose that C contains neighbours of both u and v and C contains no dense vertices. Lemma 7 implies that no edge in C has both endpoints in $N(u)$ or $N(v)$. If u and v have a common neighbour in C, then this implies $|V(C)| = 1$. Thus, consider that u and v have no common neighbour in C. Lemma 4 implies that any vertex in C is adjacent to either u or v. This fact together with Lemma 1 implies that C has at most four vertices. A more carefull analysis shows that C can have at most three vertices. If C has four vertices, then we can build two cycles in G that share an edge by pairing the paths in C between pairs of vertices together with u and v. To be more precise, suppose that C has four vertices u', u'', v' and v'' where $u', u'' \in N(u)$ and $v', v'' \in N(v)$. Since C is connected, it must have at least three edges. Moreover, C is bipartite as $\{u', u''\}$ and $\{v', v''\}$ are stable and disjoint. Therefore, there is at least one vertex in C adjacent to both vertices in the other side of the bipartition. Without loss of generality, assume that u' is such a vertex. Also without loss of generality, assume that u'' is adjacent to v''. We get a contradiction as the cycles (u', v'', u'', u) and (u', v'', v, v') share the edge $u'v''$. Thus, C has at most three vertices. By contradiction, suppose that e is an edge of C with endpoints u' and v' such that $u' \in N(u)$ and $v' \in N(v)$. As before, note that $m(G-e) = m$ which implies that $\chi_b(G-e) = m$. Let ψ be a b-colouring of $G - e$ with m colours. Since ψ cannot be a b-colouring of G, then $\psi(u') = \psi(v') = c$. Let w be the b-vertex coloured c. The fact that C has at most three vertices implies that either u or v has only one neighbour in C. Without loss of generality, suppose that v has only one neighbour r in C. Let Q be the component of $G - u$ that contains v. Let (B_v, B_u) be the partition of $V(G)$ defined by $B_v = V(Q) \setminus N(u)$ and $B_u = V(G) \setminus B_v$. If w is in B_u, then $\psi(B_v, c \leftrightarrow \psi(u))$ is a b-colouring of G with m colours. Thus, consider that w is in B_v. If u is not adjacent to a vertex in C with colour $\psi(v)$, then $\psi(B_v, c \leftrightarrow \psi(v))$ is a b-colouring of G with m colours. Thus, u has another neighbour in C with colour $\psi(v)$. Note that in this case, r is adjacent to v and to two neighbours in C which implies $d(r) = 3$. If there is a colour c' not in the set $\{c, \psi(v), \psi(u)\}$ whose b-vertex is in B_v, then $\psi(B_v, c \leftrightarrow c')$ is a b-colouring of G with m colours. Thus, consider that the only b-vertices in B_v are v and w. Let $B_v^- = B_v \setminus \{r\}$. If there is a component of $G[B_v^-] - v$ that does not contain w, then Lemma 5 tells us that such a component contains a single vertex ℓ. In this case, $\psi(N(v), c \leftrightarrow \psi(\ell))$ is a b-colouring of G with m colours. If no such component exists, then Lemma 1 tells us that $d(v) = 3$ and $m = 4$. This is a contradiction as we assumed there were no dense vertices in C and $d(r) = 3$. Therefore, u and v have at least one common neighbour in C which, in turn, implies the lemma. □

Proof of Theorem 3. Let G be an anomalous minimal m-defective cactus and let e be an edge between u and v such that $u, v \notin D(G)$. Let $H = G - e$. Since $m(H) = m(G) = m$ and G is minimal m-defective, then $\chi_b(H) = m$. Thus, suppose that ψ is a b-colouring of H with m colours. Note that Lemma 6 implies that each colour class in ψ has precisely one dense vertex and this vertex is a b-vertex. If $\psi(u) \neq \psi(v)$, then we get a contradiction as ψ is also a b-colouring of G. Thus, assume that $\psi(u) = \psi(v)$ and let w be the b-vertex of this colour class. Throughout the proof of this theorem, we build a contradiction by constructing a b-colouring φ from ψ with m colours such that $\varphi(u) \neq \varphi(v)$.

We first consider the case in which u and v are in two different components of H. Let C_v be the vertex set of the component of v and, without loss of generality, consider that w is not in C_v. Suppose that there is a colour c other than $\psi(w)$ and whose b-vertex is not in C_v. We get φ as $\psi(C_v, c \leftrightarrow \psi(w))$ to get a contradiction. If there is no such colour c, then u is not adjacent to any b-vertex. Since u has degree less than $m-1$ in G, we can change its colour to a colour it is not adjacent to get φ.

We now assume that u and v are in the same component and there is a unique path P between them in H. For $z \in V(P)$, let C_z be the vertex set of the component of z in $H - E(P)$ and $R_z = V(H) \setminus C_z$ be the remaining vertices. Without loss of generality, suppose that the distance from u to w is not greater than the distance from v to w in H, i.e., $d_H(u, w) \leq d_H(v, w)$. Note that this implies that w is not in C_v. Let y be a b-vertex adjacent to u according to Lemma 4. Note that v has only one neighbour p in R_v by the uniqueness of P.

We consider if p is a b-vertex or not.

First, consider that p is not a b-vertex. If there are at least two b-vertices in R_v other than w, at least one is from a colour c diferent than $\psi(p)$. We get φ as $\psi(R_v, c \leftrightarrow \psi(w))$. Thus, the only b-vertices in R_v are w and y. Observe that, if $\psi(p) \neq \psi(y)$, then $\varphi = \psi(R_v, \psi(w) \leftrightarrow \psi(y))$ is a b-colouring of G. Therefore, consider that $\psi(p) = \psi(y)$. Let t be the neighbour of p in $P - v$. We consider if t is a b-vertex or not. If t is not a b-vertex, choose a colour class c not in the set $\{\psi(w), \psi(y), \psi(t)\}$. One such colour class exists as $m \geq 4$. We also have that the b-vertex coloured c is not in C_p as it must be in C_v. Note that $\sigma = \psi(C_p, c \leftrightarrow \psi(y))$ is a b-colouring of H such that $\sigma(p) \neq \sigma(y)$. Thus, we get φ from σ such that v is not in the same colour class as u as mentioned before. Now consider that t is a b-vertex. Since the only two b-vertices in R_v are w and y, t is adjacent to p and $\psi(p) = \psi(y)$, then $t = w$. Let z be the neighbour of w in $P - p$ and c be a colour not in the set $\{\psi(w), \psi(z), \psi(y)\}$. One such colour class exists as $m \geq 4$. We also have that $B = C_w \cup C_p \cup C_v$ contains all b-vertices other than y. Thus, the b-vertex coloured c is in B. We get a contradiction from $\varphi = \psi(B, c \leftrightarrow \psi(w))$. Observe that if $z = y$, then it is adjacent to u and φ is a b-colouring as y is a b-vertex in φ.

Now, consider that p is a b-vertex. Suppose that p is adjacent to a vertex with colour $\psi(w)$ in R_v. If there is a colour c not in the set $\{\psi(w), \psi(p)\}$ whose b-vertex is in R_v, then $\varphi = \psi(R_v, c \leftrightarrow \psi(w))$ is a b-colouring of G. If no such colour class S exists, then w and y are the only b-vertices in R_v, as $y \in R_v$. Moreover, $y = p$, as p is adjacent to v and cannot be in S_w. Since u is adjacent to y and cannot be adjacent to w, then the only b-vertex adjacent to u is y and y is also adjacent to v. Since $d_H(u) \leq m - 3$, we get φ by changing the colour of u in ψ to some colour it is not adjacent to get a contradiction.

Now, we consider the case that p is a b-vertex but it is not adjacent to a vertex with colour $\psi(w)$ in R_v. In particular, p is not adjacent to u. Let x be the closest vertex to v in $P - v$ that is not a b-vertex. Since u is not a b-vertex, such a vertex x exists. Let P' be the unique path from x to v and let $B_v = \cup_{z \in V(P') \setminus \{x\}} C_z$ and $B_u = V(H) \setminus B_v$. We now consider the possibilities of whether w is in B_v or B_u and whether x equals to u or not.

First, suppose that $w \in B_u$ and $u = x$. Let ℓ be the neighbour of u in P. If $\ell \neq y$, then $\varphi = \psi(B_u \cup C_\ell, \psi(w) \leftrightarrow \psi(y))$ is a b-colouring of G. Thus suppose that $\ell = y$ and let z be the neighbour of y in $P - x$. If $z \neq p$, then we get a contradiction from $\varphi = \psi(B_u \cup C_y \cup C_z, \psi(w) \leftrightarrow \psi(y))$. Thus, consider that y is adjacent to p. If there is a colour c not in $\{\psi(w), \psi(y)\}$ whose b-vertex is in $B_u \cup C_y$, then we get a contradiction from $\varphi = \psi(B_u \cup C_y, c \leftrightarrow \psi(w))$. Thus, consider that no such colour c exists and the only b-vertex adjacent to u is y. Since $d_H(u) \leq m - 3$ and u is adjacent to y, then there is a colour c not in $\{\psi(w), \psi(y), \psi(p)\}$ such that u has no neighbour coloured c. We know that the b-vertex coloured c is not in $B_u \cup C_y$. Therefore, we get a contradiction from $\varphi = \psi(C_y \cup \{u\}, c \leftrightarrow \psi(w))$.

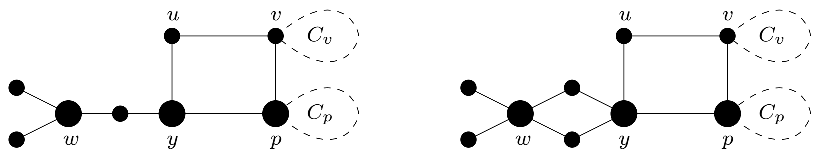
Now, suppose that $w \in B_u$ and $x \neq u$. If there is a colour c not in the set $\{\psi(w), \psi(x)\}$ whose b-vertex is in B_u, then we get a contradiction from $\varphi = \psi(B_u, c \leftrightarrow \psi(w))$. If no such colour c exists, then $\psi(x) = \psi(y)$, as $y \in B_u$. If $w \in C_x$, then we get a contradiction from $\varphi = \psi(B_u \setminus C_x, \psi(w) \leftrightarrow \psi(p))$. Let z be the neighbour of x in B_v. If $z \neq p$, then we get a contradiction from $\varphi = \psi(B_u \cup C_z, \psi(w) \leftrightarrow \psi(y))$. Thus, let's consider that p is adjacent to x, $w \in B_u \setminus C_x$, w and y are the only two b-vertices in B_u and $\psi(x) = \psi(y)$. Let

r be the neighbour of x in $P-p$. If $\psi(r) \neq \psi(w)$, then we know that r is not a b-vertex as it is adjacent to x coloured $\psi(y)$ and we get a contradiction from $\varphi = \psi(B_v \cup C_x, \psi(w) \leftrightarrow \psi(y))$. Now, consider that $\psi(r) = \psi(w)$ and we proceed by considering whether $r = u$ or $r \neq u$ and whether w is in C_r or not. If $r \neq u$ and $w \in C_r$, then let ℓ be the neighbour of r in $P - x$ and let c be a colour not in $\{\psi(w), \psi(y), \psi(\ell)\}$. We get a contradiction from $\varphi = \psi(B_u \setminus (C_x \cup C_r), c \leftrightarrow \psi(w))$. Note that this colouring works if $\ell = y$, as u is adjacent to y. If $r \neq u$ and $w \notin C_r$, then note that r is not a b-vertex. Let ψ' be obtained from ψ by changing the colour of r to a colour it is not adjacent to. We then treat this case as if $\psi(r) \neq \psi(w)$. If $r = u$, then $y \in C_u$ as u is adjacent to x in P and x is not a b-vertex. If w is not in C_u, let c be a colour not in $\{\psi(w), \psi(y), \psi(x)\}$. We get φ by changing the colours between the vertices in $S_w \cap C_u$ and $S_c \cap C_u$. Thus, consider that w and y are in C_u. Let H_w be the component that contains w in $H[C_u] - y$. From Lemma 1, y has at most two neighbours in H_w. Since u is adjacent to x and y, $\psi(x) = \psi(y)$ and $d_H(u) \leq m - 3$, then u is not adjacent to at least three colour classes different from $\psi(w)$. Therefore, there is a colour c other than $\psi(w)$ such that u has no neighbour coloured c and y has no neighbour coloured c in H_w. Thus, φ is obtained from $\psi(C_u \setminus V(H_w), c \leftrightarrow \psi(w))$ by colouring u with colour c.

Now, suppose that $w \in B_v$ and $x \neq u$. Let z be the neighbour of x in B_v. If there is a colour c not in $\{\psi(x), \psi(w), \psi(z)\}$ whose b-vertex is in B_v, then we get a contradiction from $\varphi = \psi(B_v, c \leftrightarrow \psi(w))$. Note that if $z \neq p$ and $\psi(x) \neq \psi(p)$, then with $c = \psi(p)$ we get $c \notin \{\psi(x), \psi(w), \psi(z)\}$. Thus, if no such colour c exists, then $z = p$ or $\psi(x) = \psi(p)$. We know that $z \neq p$, as $d_H(u, w) \leq d_H(v, w)$. Therefore, consider that $\psi(x) = \psi(p)$. If $z \neq w$, then we get a contradiction from $\varphi = \psi(B_v, \psi(w) \leftrightarrow \psi(z))$. If $z = w$, then there is a vertex r of P' such that $\psi(r) \notin \{\psi(w), \psi(p)\}$, as p is not adjacent to w. As before, we get a contradiction from $\varphi = \psi(B_v, \psi(w) \leftrightarrow \psi(r))$.

Finally, suppose that $w \in B_v$ and $x = u$. In this case, we show properties of G that imply that G is isomorphic to a graph in Figure 2 or Figure 3. Let ℓ be the neighbour of u in P and t be the neighbour of ℓ in $P - u$. If $w \notin C_\ell$, then $t \neq p$ as $d_H(u, w) \leq d_H(v, w)$. In this case, we get a contradiction from $\varphi = \psi(B_u \cup C_\ell, \psi(w) \leftrightarrow \psi(p))$ as ℓ is adjacent to a vertex with colour $\psi(p)$ in C_ℓ. If $w \in C_\ell$ and $\ell \neq y$, then we get a contradiction from $\varphi = \psi(B_u \cup C_\ell, \psi(w) \leftrightarrow \psi(y))$. If $w \in C_\ell$, $\ell = y$ and $t \neq p$, then we get a contradiction from $\varphi = \psi(B_u \cup C_\ell \cup C_t, \psi(w) \leftrightarrow \psi(y))$. Therefore, we get $y = \ell$, $w \in C_y$ and $t = p$. If there is a colour c not in $\{\psi(w), \psi(p), \psi(y)\}$ whose b-vertex is in $B_u \cup C_y$, then we get a contradiction from $\varphi = \psi(C_v \cup C_p, c \leftrightarrow \psi(w))$. Thus, there is no b-vertex in B_u and the only b-vertices in C_y are y and w. Using Lemma 4, we get that $B_u = \{u\}$. If $d_G(y) \geq m$, then note that $m(G - u) = m(G) = m$ and, therefore, $\chi_b(G - u) = m$. If σ is a b-colouring of $G - u$ with m colours, then we can extend σ to a b-colouring of G by giving a colour to u not in $\{\sigma(y), \sigma(v)\}$. Therefore, we get $d_G(y) = m - 1$. Note that this implies that y is not adjacent to a vertex with colour $\psi(w)$ in C_y as all of its neighbours must recieve distinct colours and y is adjacent to u. In particular, y is not adjacent to w. If there is a component Q of $G[C_y] - y$ that does not contain w, then Q contains no dense vertices. Lemma 5 tells us that $|V(Q)| = 1$. Let r be the neighbour of y in Q. We get φ from ψ by exchanging the colour between r and u. Thus, suppose that $G[C_y] - y$ is connected. Lemma 1 implies that y has at most two neighbours in $G[C_y]$. Therefore, $d(y) \leq 4$ which implies that $m \leq 5$

as $y \in D(G)$. If R is a component of $G[C_y] - w$ that does not contain y, then Lemma 5 implies that $|V(R)| = 1$. Lemma 5 also tells us that $d(w) = m - 1$ as $d(w) \geq 3$ together with Lemma 1 implies that at least one such R exists and w has at least one neighbour of degree one. Now, we apply Lemma 8 to conclude that the neighbours of y not in $\{u, p\}$ are also neighbours of w. Therefore, to get the structure of G, we have to describe the behaviour of C_v and C_p. Moreover, G has the structure of Figure 4(a) if $m = 4$ and of Figure 4(b) if $m = 5$. We now consider the properties of the vertices in $D(G) \setminus \{w, y, p\}$. Note that $D(G) \setminus \{w, y, p\}$ has either one or two vertices depending on whether m equals to four or five. Moreover, they are either in C_p or in C_v.



(a) Scheme for G when $m = 4$. (b) Scheme for G when $m = 5$.

Figure 4: Schemes for an anomalous cactus.

First, consider that there are no dense vertices in $C_p \setminus \{p\}$. In this case, v has at least one neighbour in C_v. This implies that $d_G(v) \geq 3$ and thus $m \geq 5$ as $v \notin D(G)$. Moreover, $m \leq 5$ implies $d_G(v) \leq 3$. Therefore, v has precisely one neighbour in C_v. Now, Lemma 5 tells us that $d_G(p) = m - 1 = 4$ and that p has two neighbours in C_p with degree one. Let r_1 and r_2 be the two dense vertices in $D(G) \setminus \{w, y, p\}$. Note that since p is adjacent to y and to v with $\psi(v) = \psi(w)$, then the two neighbours of p in C_p are coloured $\psi(r_1)$ and $\psi(r_2)$. We claim that $d_G(r_1) = d_G(r_2) = m - 1 = 4$, r_1 and r_2 are adjacent and v is adjacent to a vertex in $\{r_1, r_2\}$. To see that $d_G(r_1) = d_G(r_2) = m - 1 = 4$, note that $d_G(r_1) \geq 4$ together with Lemma 1 implies that there is a component of $G - r_1$ that contains neither v nor r_2. Lemma 5 with a symmetric argument for r_2 concludes that $d_G(r_1) = d_G(r_2) = m - 1 = 4$. Consider the two colourings $\sigma_1 = \psi(N(p), \psi(r_1) \leftrightarrow \psi(v))$ and $\sigma_2 = \psi(N(p), \psi(r_2) \leftrightarrow \psi(v))$. If v is not adjacent to a vertex in $\{r_1, r_2\}$, then either σ_1 or σ_2 is a b-colouring of G, depending on the colour of the neighbour of v in C_v. We pick φ as this b-colouring. Without loss of generality, say that v is adjacent to r_1. If r_1 is not adjacent to r_2, then Lemmas 5 and Lemma 8 tell us that the neighbours of r_1 are v, common neighbours of r_1 and r_2 or vertices with degree one. Thus, the neighbour of r_1 with colour $\psi(r_2)$ has degree one. We get a contradiction from $\varphi = \psi(N(r_1), \psi(v) \leftrightarrow \psi(r_2))$. This concludes our claim and Lemma 7 implies that the cactus graphs that satisfy these properties are isomorphic to the anomalous graphs in Figure 3(a) or Figure 3(b).

Now, consider there is precisely one dense vertex in $C_p \setminus \{p\}$. Let r be the dense vertex in $C_p \setminus \{p\}$. We consider two cases based on whether $m = 4$ or $m = 5$. If $m = 4$, then vertices not in $D(G)$ have degree at most two in G. This implies that $d_G(v) = 2$ as it is adjacent to p and u and that $C_v = \{v\}$. We claim that p is adjacent to r and that $d_G(p) = m - 1 = 3$. Note that p has a neighbour with colour $\psi(r)$ in C_p. By contradiction, suppose they are not adjacent and ℓ is the neighbour of p with colour $\psi(r)$. Let Q be the component that contains ℓ in $G-p$. Note that $V(Q) \subseteq C_p$. Since ℓ cannot be adjacent to r, Lemma 8 tells us that r is not in Q. Thus, Lemma 5 implies ℓ is the unique vertex in Q. We get a contradiction with φ obtained from ψ by exchanging the colour between v and ℓ. Therefore, p and r are adjacent. If $d_G(p) \geq m$, then note that $m(G-v) = m(G) = m$. Therefore, $\chi_b(G-v) = m$ as G is minimal m-defective. If σ is a b-colouring of $G-v$ with m colours, then we can get a b-colouring of G from σ by giving a colour to v not used in its neighbourhood to get a contradiction. Therefore, $d_G(p) = m-1 = 3$. Lemma 7 concludes that any cactus graphs that satisfies these properties is isomorphic to the anomalous graph in Figure 2. Now, consider the case $m = 5$ and there is a dense vertex in C_v, namely r'. We claim that v is adjacent to r', $d_G(p) = m-1 = 4$ and p is adjacent to r. If v is not adjacent to r', then Lemma 4 implies that all vertices in $C_v \setminus \{v\}$ are adjacent to r'. Therefore, the neighbour of v in C_v is not coloured $\psi(r')$. We get a contradiction from $\varphi = \psi(C_p \cup \{v\}, \psi(w) \leftrightarrow \psi(r'))$. Thus, v is adjacent to r'. If p is not adjacent to r, then Lemma 5 and Lemma 8 imply that the neighbours of p in C_p either have degree one or they have degree two and are common neighbours between p and r. Therefore, the neighbour ℓ of p with colour $\psi(r)$ has degree one. We get a contradiction from $\varphi = \psi(C_v \cup \{\ell\}, \psi(w) \leftrightarrow \psi(r))$. Also, if $d_G(p) \geq m$, then Lemma 5 implies that all neighbours of p is C_p are in a component of $G-p$ that contains r. Thus, Lemma 8 implies that the neighbours of p in C_p are either r or adjacent to r. Since $d_G(p) \geq m = 5$, then p has two neighbours in C_p other than r. Each such neighbour defines a triangle together with p and r and we get a contradiction as these two triangles share the edge pr. Therefore, the claim is valid and any cactus graph that satisfies these properties is isomorphic to the anomalous graph in Figure 3(c) or Figure 3(d).

Finally, consider there are two dense vertices in $C_p \setminus \{p\}$ other than p. Note that this implies $m = 5$. Moreover, Lemma 4 implies $C_v = \{v\}$. Let r_1 and r_2 be the two dense vertices in $D(G) \setminus \{w, y, p\}$. We claim that $d_G(p) = d_G(r_1) = d_G(r_2) = m-1 = 4$. To see that $d_G(r_1) = d_G(r_2) = m-1 = 4$, note that, if $d_G(r_1) \geq m = 5$, then Lemma 1 implies that there is a component of $G - r_1$ that contains neither p nor r_2. Lemma 5 together with a symmetric argument for r_2 concludes that $d_G(r_1) = d_G(r_2) = m-1 = 4$. If $d_G(p) \geq m = 5$, then $m(G-v) = m(G)$. Therefore, $\chi_b(G-v) = m$, as G is minimal m-defective. We get a contradiction as a b-colouring of $G-v$ with m colours can be extended to G by giving a colour not used in its neighbourhood to v. This is possible as $d_G(v) = 2$. Thus, $d_G(p) = 4$. From now on, let $\psi' = \psi(C_p, \psi(w) \leftrightarrow \psi(y))$ and note that ψ' is also a b-colouring of H with m colours. Now, we claim that at least one neighbour of p in C_p is a dense vertex. By contradiction, suppose otherwise. Note now that p has two neighbours in C_p with colours $\psi(r_1)$ and $\psi(r_2)$. If there is more than one component in $G[C_p] - p$, then let Q be one such component that does not contain r_2. Note that p has only one neighbour ℓ in Q, as its other neighbour in C_p must be in a component that contains r_2. Lemma 8 tells us that, if $\psi(\ell) = \psi(r_1)$, then r_1 is not in Q. Thus, we get a contradiction from $\varphi = \psi(V(Q) \cup \{v\}, \psi(w) \leftrightarrow \psi(\ell))$. Therefore, $G[C_p] - p$ is connected and contains both r_1 and r_2. Now, we claim that there is a path from p to r_2 that avoids r_1. To see this, suppose otherwise. Lemma 8 applied to $G-p-r_1$ tells us that the neighbours of p in C_p are also neighbours of r_1. A contradiction as p has a neighbour with colour $\psi(r_1)$ in C_p. Similarly, there is a path from p to r_1 that avoids r_2. We now claim that $G[C_p] - p - r_1$ has only one component with neighbours of p. To see this, suppose otherwise. Thus, at least one component of $G[C_p]-p-r_1$ has neighbours of both r_1 and p but does not contain r_2. From Lemma 8 applied to this component, we get that p and r_1 have a common neighbour. Note that this common neighbour must get colour $\psi(r_2)$. Moreover, Lemma 1 implies that r_1 has at most two neighbours in the component of $G-r_1$ that contains r_2. One of these neighbours is a common neighbour with p. Therefore, r_1 has two neighbours of degree one by Lemma 5. Note that r_1 has a neighbour with degree one with the same colour as w in either ψ or ψ'. Thus, assume r_1 has a neighbour of degree one with colour $\psi(w)$. We get a contradiction from $\varphi = \psi(N(r_1) \cup \{v\}, \psi(w) \leftrightarrow \psi(r_2))$. With the same argument, we also get that $G[C_p]-p-r_2$ has only one component with neighbours of p. Therefore, neither r_1 nor r_2 are in the unique path $\hat{P}$ between the neighbours of p in $G[C_p]-p$ as this would imply either $G[C_p]-p-r_1$ or $G[C_p]-p-r_2$ would have two components with neighbours of p. Lemma 7 tells us that $\hat{P}$ has at least one internal vertex and Lemma 4 tells us that such an internal vertex is adjacent to either r_1 or r_2. Note that since $\hat{P}$ has no vertices in $D(G)$ and $m=5$ all vertices in $V(\hat{P})$ have degree exactly three with two neighbours in $\hat{P}$ and one neighbour in $\{r_1, r_2\}$. Therefore, let z be an internal vertex of $\hat{P}$ and say it's adjacent to r_1. Let Q be the component of $G-z$ that contains r_1 and note that r_2 is not in Q as this would imply there is no path from p to r_2 that avoids r_1. As before, assume that $\psi(z) \neq \psi(w)$ as this happens in either ψ or ψ'. We get a contradiction from $\psi(\{v\} \cup C_p \setminus V(Q), \psi(w) \leftrightarrow \psi(r_1))$. Thus, p has at least one dense neighbour in C_p. If p has two dense neighbours in C_p, then Lemma 7 concludes that any cactus graph with these properties is isomorphic to the anomalous graphs in Figure 3(e), Figure 3(f) or Figure 3(g). Now, assume that p is adjacent to r_1 but not to r_2. Let ℓ be the neighbour of p in $C_p \setminus D(G)$, i.e., the neighbour of p in C_p that is not r_1. We claim that $d(\ell)=2$ and ℓ is adjacent to p and r_1. To see this, first note that there is a unique path $\hat{P}$ from ℓ to r_1 that avoids p. By contradiction, suppose that $d(\ell) \geq 3$ which implies that ℓ has a neighbour z which is not p and is not in $\hat{P}$. We know that $z \neq r_2$ as $\psi(\ell) = \psi(r_2)$ which implies that $z \notin D(G)$. Now, Lemma 4 implies that z is adjacent to r_2. Assume that z does not have the same colour as w as this is true in either ψ or ψ'. If Q is the component of $G-\ell$ that contains z, then we get a contradiction from $\varphi = \psi(\{v\} \cup C_p \setminus V(Q), \psi(w) \leftrightarrow \psi(r_2))$. Therefore, $d(\ell)=2$. Now, let z' be the neighbour of ℓ in $\hat{P}$ and suppose that $z' \neq r_1$. We can assume that z' does not have the same colour as w as this is true in either ψ or ψ'. Therefore, we get a contradiction from $\varphi = \psi(N(p), \psi(w) \leftrightarrow \psi(r_2))$ which proves our recent claim. Note that this implies that r_1 disconnects p from r_2, i.e., p and r_2 are in two different components of $G-r_1$. Moreover, the neighbours of r_2 with colours $\psi(w)$ and $\psi(y)$ are not in the component of $G-r_1$ that contains p. We claim that these two neighbours of r_1 have degree two and are common neighbours between r_1 and r_2. Lemma 5 together with Lemma 8 imply that the neighbours of r_1 with colours $\psi(w)$ and $\psi(y)$ either have degree one or have degree two and are adjacent to r_2. Thus, suppose that r_1 is adjacent to z with degree one. Note that $\psi(z) \in \{\psi(w), \psi(y)\}$. We can assume that $\psi(z) = \psi(w)$ as this is true in ψ or ψ'. Thus, we get a contradiction from $\varphi = \psi(N(r_1) \cup \{v\}, \psi(w) \leftrightarrow \psi(r_2))$. With this claim together with Lemma 7, we know that any cactus graph that satisfies these properties is isomorphic to the anomalous graph in Figure 3(h). $\square$

A proper subgraph H of a minimal m-defective graph G is said to be a spoon of G if $\chi_b(H) = m(H) = m$. If G is minimal m-defective and contains a spoon, we say that G is a spoonie.

Lemma 9. *Let G be a minimal m-defective cactus with $m \geq 4$. If G is a spoonie, then there exists an edge e such that $G - e$ is a spoon.*

Proof. Let H be a spoon of G and $H' = (V(G), E(H))$. Since $V(G) \setminus V(H)$ is stable in H', we can extend a b-colouring ψ of H with m colours into a b-colouring of H' by giving an arbitrary colouring to the uncoloured vertices from the colours in ψ. Thus H' is a spoon. Let e be an edge of G that is not in H' and let $H'' = G - e$. Since H'' is a supergraph of H', then $m(H'') \geq m(H') = m$ and since it is a subgraph of G, then $m(H'') \leq m(G) = m$. This implies that H'' is a spoon of G as it is a proper subgraph and G is minimal m-defective. □

In the lemma that follows, we prove that, other than anomalous graphs, "there is no spoon".

Lemma 10. *Let G be a minimal m-defective cactus with $m \geq 4$. If G is not anomalous and H is a proper subgraph of G, then $m(H) < m$.*

Proof. Suppose that G is a spoonie and is not anomalous. By Lemma 9, let $e = uv \in E(G)$ be an edge such that $H = G - e$ is a spoon of G. Since $|D(G)| = m$ and $\chi_b(H) = m$, we have that $D(H) = D(G)$ and the dense vertices in $D(H)$ are the unique b-vertices in any b-colouring of H with m colours. Let ψ be a b-colouring of H with m colours and note that every colour class in ψ has precisely one b-vertex. Observe that we get a contradiction if we can build a b-colouring φ of H from ψ such that $\varphi(u) \neq \varphi(v)$ as this is also a b-colouring of G. From Theorem 3, we know that at least one endpoint of e is a dense vertex. If u and v are both dense vertices, then we get a contradiction from $\varphi = \psi$ as $\psi(u) \neq \psi(v)$. Thus, without loss of generality, suppose that u is a dense vertex and v is not. We know that $d_G(u) \geq m$ as $|D(G)| = m$ and $m(H) = m$. Note that, if w is a neighbour of u in G and w is not a dense vertex, then $d_G(w) > 1$ from Lemma 5.

We first consider that u and v are in different components of H. Let C_v be the set of vertices in the component of v and R_v be the remaining vertices. If there is a b-vertex w different from u in R_v, then we get a contradiction from $\varphi = \psi(R_v, \psi(w) \leftrightarrow \psi(u))$. If there is no b-vertex in R_v other than u, let w be a neighbour of u in R_v. At least one such neighbour exists as $d_H(u) \geq m-1 \geq 3$. By Theorem 3, we have $d_H(w) = d_G(w) = 1$, a contradiction.

We now consider the case that u and v are in the same component and there is a unique path P between them in H. Let z be the neighbour of v in P and p be the neighbour of z in $P - v$. Note that Theorem 3 implies $z \in D(G)$ as it is a neighbour of v. Let C_u be the vertex set of the component of u in $H \setminus E(P)$ and C'_u be the vertex set of the component of u in $H - zp$. If u is the only b-vertex in C'_u, then u has at least one neighbour different from p and it has degree one in G by Theorem 3, a contradiction. If C'_u has a b-vertex w not coloured $\psi(u)$ or $\psi(p)$, then we obtain a contradiction from $\varphi = \psi(C'_u, \psi(w) \leftrightarrow \psi(u))$. Note that this works even if $p = u$, as this case would imply z is adjacent to u and v with $\psi(u) = \psi(v)$. If $d_G(u) \geq 5$, then u has at least three neighbours in C_u. The fact that all neighbours of u in H have degree at least two together with Lemma 1 and Theorem 3 tells us that there are at least two b-vertices in C_u

other than u. A contradiction as at least one of them is not coloured $\psi(u)$ or $\psi(p)$. Thus we can assume that $d_G(u) = 4$. Note also that there is precisely one dense vertex w in C_u other than u and $\psi(w) = \psi(p)$. Therefore, p is not a dense vertex as w cannot equal p. Note also that the fact that the only dense vertices in C'_u are u and w together with the fact that G is not anomalous implies that u is adjacent to p. Since p and v are not dense vertices, they have degree at most two in G. Since they are neighbours of z and u in G, then $d_G(v) = d_G(p) = 2$ and $\{u, p, z, v\}$ induces a cycle in G. The vertex u has degree three in H and is adjacent to p with $\psi(p) = \psi(w)$, so it cannot be adjacent to w. Thus, there are two vertices x and y of degree two adjacent to u and w forming another cycle induced by $\{u, x, w, y\}$ in G. Let q be the dense vertex of $D(G)$ not in the set $\{u, w, z\}$. If z is not adjacent to q, then let r be the neighbour of z with colour $\psi(q)$. It is not a neighbour of q, u or w, so it has degree one. We get φ from ψ by exchanging the colours between v and r. Now consider the case that z is adjacent to q. Let E_z be the set of edges with one endpoint in z and are not incident to v, p or q. Let H' be obtained from G by removing the edge ux and the edges in E_z. Observe that $m(H') = m = 4$, so let's consider a b-colouring σ of H' with m colours. The vertices v and p are adjacent to z and u and, since $d_{H'}(z) = 3$ and z is adjacent to q, they can only receive colour $\sigma(w)$. Since they are both adjacent to z, they cannot have the same colour. Thus, H' is m-defective and a proper subgraph of G. A contradiction as G is minimal m-defective. □

Proof of Theorem 2. Let $m \geq 4$ and G be a minimal m-defective cactus. If G is anomalous, then G is isomorphic to a graph in Figure 2 or in Figure 3 which satisty the theorem. Now, suppose G is not anomalous. From Lemma 6, we know that $|D(G)| = m$. If G has a vertex u of degree at least m, then u has a neighbour w not in $D(G)$. We get a contradiction from Lemma 10 as $m(G - uw) = m$. □

APÊNDICE E – OUTROS RESULTADOS OBTIDOS

Transversais em árvores

V. Campos V. Chvátal L. Devroye P. Taslakian

Uma transversal em uma árvore enraizada é um conjunto de nós que intersecta cada caminho entre a raiz e uma folha. Denote por $c(T,k)$ o número de transversais de tamanho k em uma árvore T. Dadas árvores T e T' com n nós, utilizamos $T \succ T'$ para indicar que $c(T,k) \geq c(T',k)$ para todo k e que $c(T,k) > c(T',k)$ para pelo menos um k. Neste caso, dizemos que T' é mais robusta do que T. Mostramos operações em árvores que a transformam em uma árvore mais robusta. Utilizamos estas operações para mostrar a estrutura da árvore d-ária mais robusta para qualquer d fixo. Além disto, estudamos a probabilidade $\xi(T,p)$ de obter uma transversal quando escolhemos cada nó de T de forma independente com probabilidade p. Mostramos limites superior e inferior apertados para $\xi(T,p)$ e mostramos que uma árvore binária de busca satisfaz o limite superior assintoticamente.

O artigo referente a estes resultados foi aceito para publicação no *Journal of Graph Theory* [1].

Referências

[1] V. Campos, V. Chvátal, L. Devroye, and P. Taslakian. Transversals in trees. *Journal of Graph Theory*, 2011. Aceito para publicação.

Número de escolha de grafos quase planares

V. Campos F. Havet

Uma *coloração de vértices* de um grafo G é uma atribuição de cores aos vértices de G de forma que vértices adjacentes recebam cores distintas. Dadas listas $L(v)$ associadas a cada vértice $v \in V(G)$, uma *coloração por listas* de G é uma coloração de vértices de forma que cada vértice v recebe uma cor de sua lista $L(v)$. Caso tal coloração exista, dizemos que G é *colorível pelas listas* L. O *número de escolha* $ch(G)$ é o menor inteiro positivo k tal que, para quaisquer listas L com $|L(v)| \geq k$ para todo $v \in V(G)$, G é colorível pelas listas L. O *número de cruzamento* $cr(G)$ é o menor inteiro k tal que G pode ser desenhado no plano com k cruzamentos de arestas. Se $cr(G) = 0$, dizemos que G é *planar*. Resolvendo uma conjectura de P. Erdős aberta há quase 20 anos, C. Thomassen mostrou que se G é planar, então $ch(G) \leq 5$ [2]. Melhorando o resultado de C. Thomassen, mostramos que se G tem uma aresta tal que $G - e$ é planar, então $ch(G) \leq 5$. Também mostramos que se $cr(G) \leq 2$, então $ch(G) \leq 5$. As provas são construtivas e dão origem a algoritmos polinomiais para colorir G por listas.

O artigo referente a estes resultados foi submetido para publicação no *Journal of Graph Theory* [1].

Referências

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[2] C. Thomassen. Every planar graph is 5-choosable. *Journal of Combinatorial Theory, Series B*, 62:180–181, 1994.

Sobre o número de hull de algumas classes de grafos

J. Araujo V. Campos F. Giroire N. Nisse
L. Sampaio R. Soares

Dado um grafo G, o *intervalo fechado* de dois vértices u e v, denotado por $I(u, v)$, é o conjunto de vértices de G que pertencem a algum caminho mínimo entre u e v. Para $S \subseteq V(G)$, seja $I(S) = \cup_{u,v \in S} I(u, v)$. Dizemos que S é um *conjunto convexo* se $I(S) = S$. O fecho convexo $I_h(S)$ de um subconjunto de vértices S é o menor conjunto convexo que contém S. Dizemos que S é um *conjunto hull* se $I_h(S) = V(G)$. A cardinalidade de um conjunto hull mínimo de G é o número de hull de G, denotado por $hn(G)$. Mostramos que decidir se $hn(G) \leq k$ é um problema NP-completo para a classe de grafos bipartidos e provamos que $hn(G)$ pode ser calculado em tempo polinomial para complementos de bipartidos, grafos $(q, q-4)$ e grafos cacto. Também apresentamos alguns limites superiores para o número de hull de grafos em geral, grafos livres de triângulos e grafos regulares.

O artigo referente a estes resultados foi publicado no *Eurocomb 2011* [1].

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Problemas de coloração restritos em grafos com poucos P_4s

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R. Sampaio

Obtemos algoritmos polinomiais para determinar o número acíclico cromático, o número estrela cromático e o número harmônico cromático de grafos P_4-tidy e grafos $(q, q-4)$, para qualquer q fixo. Estas classes incluem cografos, grafos P_4-esparsos e grafos P_4-lite. Também obtemos um algoritmo polinomial para determinar o número de Grundy de grafos $(q, q-4)$. Todos estes problemas de coloração são NP-difíceis para grafos em geral.

O artigo referente a estes resultados foi publicado no *LAGOS 2011* [1].

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Dois algoritmos de parâmetro fixo para o problema de cocoloração

V. Campos S. Klein R. Sampaio A. Silva

Uma (k, ℓ)-cocoloração de um grafo G é uma partição do conjunto de vértices de G em no máximo k conjuntos independentes e no máximo ℓ cliques. Dados um grafo G e inteiros k e ℓ, o *problema de cocoloração* é o problema de decidir se G tem uma (k, ℓ)-cocoloração. O menor inteiro $k + \ell$ tal que G tem uma (k, ℓ)-cocoloração é chamado de número cocromático de G. É conhecido que determinar o número cocromático é um problema NP-difícil [3]. Em 2011, Bravo et al. obtiveram um algoritmo polinomial para grafos P_4-esparsos [1]. Generalizamos este resultado para a classe de grafos $(q, q-4)$, para qualquer q fixo. Um grafo é $(q, q-4)$ se qualquer conjunto de até q vértices induz no máximo $q-4$ P_4s diferentes. Os grafos P_4-esparsos correspondem a grafos $(5, 1)$. Além disto, provamos que o problema de cocoloração é FPT quando parametrizado pela largura em árvore $tw(G)$ ou pelo parâmetro $q(G)$ definido como o menor q tal que G é um grafo $(q, q-4)$.

O artigo referente a estes resultados foi aceito para publicação no *ISAAC 2011* [2].

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